THE FOUCAULT PENDULUM

Marion and Thornton gives the standard treatment of the Foucault pendulum in Example 10.5 on pp. 398–401. However, there is an easier way to get the same result. The basic idea is to separate the problem into 2 parts: an ordinary pendulum influenced by gravity, and a Coriolis-like effect acting on the direction of motion of the pendulum.

Let $\vec{\boldsymbol{q}}$ denote the initial direction of motion of the pendulum. We are not concerned with the periodic nature of the motion, but only with the change in $\vec{\boldsymbol{q}}$ from one period to the next. With respect to an observer on the Earth we have

$$\vec{\boldsymbol{q}} = X\,\hat{\boldsymbol{\phi}} - Y\,\hat{\boldsymbol{\theta}}$$

where the signs are chosen such that X measures distance to the East and Y to the North.

Consider now the derivative of $\vec{\boldsymbol{q}}$:

$$\begin{aligned}\dot{\vec{\boldsymbol{q}}} &= \left(\dot{X}\,\hat{\boldsymbol{\phi}} - \dot{Y}\,\hat{\boldsymbol{\theta}}\right) + \left(X\,\dot{\hat{\boldsymbol{\phi}}} - Y\,\dot{\hat{\boldsymbol{\theta}}}\right)\\ &= \left(\dot{X}\,\hat{\boldsymbol{\phi}} - \dot{Y}\,\hat{\boldsymbol{\theta}}\right) - \mathit{\Omega}\cos\theta\left(X\,\hat{\boldsymbol{\theta}} + Y\,\hat{\boldsymbol{\phi}}\right) - \mathit{\Omega}\sin\theta X\hat{\boldsymbol{r}}\end{aligned}$$

Now the last term involves vertical motion, which we will ignore — gravity ensures that the pendulum moves (nearly) orthogonal to $\hat{\boldsymbol{r}}$. But gravity is the only force involved, so that the remaining terms should vanish. Equivalently, the *only* change in $\vec{\boldsymbol{q}}$ should be in the $\vec{\boldsymbol{r}}$ direction *with respect to a fixed observer*. We therefore obtain the equations

$$\begin{aligned}\dot{X} - \mathit{\Omega}\cos\theta\,Y &= 0\\ \dot{Y} + \mathit{\Omega}\cos\theta\,X &= 0\end{aligned}$$

These equations can be easily solved by introducing the variable

$$Z = X + iY$$

which results in

$$\dot{Z} + i\mathit{\Omega}\cos\theta\,Z = 0$$

This equation has the solution

$$Z(t) = Z(0)e^{-i(\mathit{\Omega}\cos\theta)t}$$

or equivalently

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} \cos(\psi t) & \sin(\psi t) \\ -\sin(\psi t) & \cos(\psi t) \end{pmatrix}\begin{pmatrix} X(0) \\ Y(0) \end{pmatrix}$$

where $\psi = \mathit{\Omega}\cos\theta = \pm\mathit{\Omega}\sin\lambda$, and where the sign depends on the hemisphere. In the Northern Hemisphere, the initial direction of the pendulum therefore rotates *clockwise* by an angle ψt.