ROTATIONS IN THE PLANE

1. INTRODUCTION

In Cartesian coordinates, the natural orthonormal basis is $\{\vec{\imath},\vec{\jmath}\}$, where $\vec{\imath} \equiv \hat{\boldsymbol{x}}$ and $\vec{\jmath} \equiv \hat{\boldsymbol{y}}$ denote the unit vectors in the x and y directions, respectively. The position vector from the origin to the point (x,y) takes the form

$$\vec{\boldsymbol{r}} = x\vec{\imath} + y\vec{\jmath}$$

Note that $\vec{\imath}$ and $\vec{\jmath}$ are constant.

A moving object has a position vector given by

$$\vec{\boldsymbol{r}}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$$

Its velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ are obtained by differentiation, resulting in

$$\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath}$$
$$\vec{\boldsymbol{a}} = \ddot{\vec{\boldsymbol{r}}} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath}$$

where dots denote differentiation with respect to t.

If we wish to describe things as seen from some point (p,q) other than the origin, all we have to do is replace $\vec{\boldsymbol{r}}$ by

$$\vec{\boldsymbol{r}}_{rel} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{R}}$$

where

$$\vec{\boldsymbol{R}} = p\vec{\imath} + q\vec{\jmath}$$

If the point is fixed ($\vec{\boldsymbol{R}} =$ constant), then this has no effect on the velocity and acceleration:

$$\vec{\boldsymbol{v}}_{rel} = \dot{\vec{\boldsymbol{r}}}_{rel} = \dot{\vec{\boldsymbol{r}}} - \dot{\vec{\boldsymbol{R}}} = \dot{\vec{\boldsymbol{r}}} = \vec{\boldsymbol{v}}$$
$$\vec{\boldsymbol{a}}_{rel} = \ddot{\vec{\boldsymbol{r}}}_{rel} = \ddot{\vec{\boldsymbol{r}}} - \ddot{\vec{\boldsymbol{R}}} = \ddot{\vec{\boldsymbol{r}}} = \vec{\boldsymbol{a}}$$

If, however, the reference point is moving ($\vec{\boldsymbol{R}} = \vec{\boldsymbol{R}}(t)$), then of course the *relative* velocity ($\vec{\boldsymbol{v}}_{rel}$) will differ from the "true" velocity ($\vec{\boldsymbol{v}}$) previously computed by the velocity ($\dot{\vec{\boldsymbol{R}}}$) of the moving reference point, and similarly for the relative acceleration.

For instance, suppose an observer undergoes constant *linear acceleration* in the x direction, so that

$$\vec{\boldsymbol{R}} = \frac{1}{2}at^2\vec{\imath}$$

Such an observer would measure relative velocity and acceleration given by

$$\vec{\boldsymbol{v}}_{rel} = \dot{\vec{\boldsymbol{r}}} - \dot{\vec{\boldsymbol{R}}} = \vec{\boldsymbol{v}} - at\vec{\imath}$$
$$\vec{\boldsymbol{a}}_{rel} = \ddot{\vec{\boldsymbol{r}}} - \ddot{\vec{\boldsymbol{R}}} = \vec{\boldsymbol{a}} - a\vec{\imath}$$

2. POLAR COORDINATES

In polar coordinates

$$
\begin{aligned}
x &= r\cos\phi \\
y &= r\sin\phi
\end{aligned}
$$

the natural orthonormal basis is $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}\}$, where

$$
\begin{aligned}
\hat{\boldsymbol{r}} &= \quad \cos\phi\,\vec{\boldsymbol{\imath}} + \sin\phi\,\vec{\boldsymbol{\jmath}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\vec{\boldsymbol{\imath}} + \cos\phi\,\vec{\boldsymbol{\jmath}}
\end{aligned}
$$

We wish to describe the *relative* position of an object with respect to an observer located at the (for now) *fixed* point $(R,\mathit{\Phi})$, *not* at the origin. How does this observer describe vectors? The natural basis is just $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}\}$ *at the location of the observer*, that is, with $\phi = \mathit{\Phi}$. Thus, the observer describes the object using the relative position vector

$$\vec{\boldsymbol{r}}_{rel}(t) = X(t)\,\hat{\boldsymbol{r}} + Y(t)\,\hat{\boldsymbol{\phi}}$$

for some functions X and Y, which points from the observer to the object. [1] The observer's own position vector, described using the same basis, is of course [2]

$$\vec{\boldsymbol{R}} = R\,\hat{\boldsymbol{r}}$$

The "true" position vector (relative to the origin) is a combination of the observer's position and the object's position relative to the observer, that is

$$\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{R}} + \vec{\boldsymbol{r}}_{rel}(t)$$

However, just as before, so long as $\vec{\boldsymbol{R}}$ is constant, the relative velocity and acceleration

$$
\begin{aligned}
\vec{\boldsymbol{v}}_{rel} &= \dot{X}\,\hat{\boldsymbol{r}} + \dot{Y}\,\hat{\boldsymbol{\phi}} \\
\vec{\boldsymbol{a}}_{rel} &= \ddot{X}\,\hat{\boldsymbol{r}} + \ddot{Y}\,\hat{\boldsymbol{\phi}}
\end{aligned}
$$

will be the same as the "true" velocity and acceleration.

3. ROTATING FRAME

Consider now a rotating observer whose position is given by

$$
\begin{aligned}
r &= R = \text{constant} \\
\phi &= \mathit{\Phi} = \mathit{\Omega} t
\end{aligned}
$$

Observers in this frame will naturally continue to use $\vec{\boldsymbol{r}}_{rel}$, $\vec{\boldsymbol{v}}_{rel}$, and $\vec{\boldsymbol{a}}_{rel}$ to describe relative motion. And they will need to take into account the fact that $\vec{\boldsymbol{R}}$ is not constant in order to compare their description to the "true" values. But there is now an additional complication, since the basis vectors themselves change with time.

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[1] The functions (X,Y) define *Cartesian* coordinates (with a particular orientation) centered at the observer's location; this is always true when working with orthonormal bases.

[2] Since the basis being used "lives" at the observer's position, the position vector does not really "live" at the origin, even though it is usually drawn that way.

a) Velocity and Acceleration

The basis vectors now take the form

$$\begin{aligned}\hat{\boldsymbol{r}} &= \quad \cos(\Omega t)\,\vec{\boldsymbol{\imath}} + \sin(\Omega t)\,\vec{\boldsymbol{\jmath}}\\ \hat{\boldsymbol{\phi}} &= -\sin(\Omega t)\,\vec{\boldsymbol{\imath}} + \cos(\Omega t)\,\vec{\boldsymbol{\jmath}}\end{aligned}$$

so that

$$\begin{aligned}\dot{\hat{\boldsymbol{r}}} &= -\Omega\sin(\Omega t)\,\vec{\boldsymbol{\imath}} + \Omega\cos(\Omega t)\,\vec{\boldsymbol{\jmath}}\\ \dot{\hat{\boldsymbol{\phi}}} &= -\Omega\cos(\Omega t)\,\vec{\boldsymbol{\imath}} - \Omega\sin(\Omega t)\,\vec{\boldsymbol{\jmath}}\end{aligned}$$

Comparing these equations with the preceding ones, we see that [3]

$$\begin{aligned}\dot{\hat{\boldsymbol{r}}} &= \quad \Omega\,\hat{\boldsymbol{\phi}}\\ \dot{\hat{\boldsymbol{\phi}}} &= -\Omega\,\hat{\boldsymbol{r}}\end{aligned}$$

We are finally ready to compare the relative and "true" velocities and accelerations. Differentiating

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{R}} + \vec{\boldsymbol{r}}_{rel} = (R+X)\,\hat{\boldsymbol{r}} + Y\,\hat{\boldsymbol{\phi}}$$

we obtain

$$\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}} = \left(\dot{X}\,\hat{\boldsymbol{r}} + \dot{Y}\,\hat{\boldsymbol{\phi}}\right) + \left((R+X)\Omega\,\hat{\boldsymbol{\phi}} - Y\Omega\,\hat{\boldsymbol{r}}\right)$$

Further differentiation yields

$$\begin{aligned}\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} &= \left(\ddot{X}\,\hat{\boldsymbol{r}} + \ddot{Y}\,\hat{\boldsymbol{\phi}}\right) + 2\left(\dot{X}\Omega\,\hat{\boldsymbol{\phi}} - \dot{Y}\Omega\,\hat{\boldsymbol{r}}\right) - \left((R+X)\Omega^2\,\hat{\boldsymbol{r}} + Y\Omega^2\,\hat{\boldsymbol{\phi}}\right)\\ &= \vec{\boldsymbol{a}}_{rel} + 2\Omega\left(\dot{X}\,\hat{\boldsymbol{\phi}} - \dot{Y}\,\hat{\boldsymbol{r}}\right) - \Omega^2\,\vec{\boldsymbol{r}}\end{aligned}$$

We therefore obtain the following equations for the relative velocity and acceleration:

$$\begin{aligned}\vec{\boldsymbol{v}}_{rel} &= \vec{\boldsymbol{v}} - \left((R+X)\Omega\,\hat{\boldsymbol{\phi}} - Y\Omega\,\hat{\boldsymbol{r}}\right)\\ \vec{\boldsymbol{a}}_{rel} &= \vec{\boldsymbol{a}} - 2\Omega\left(\dot{X}\,\hat{\boldsymbol{\phi}} - \dot{Y}\,\hat{\boldsymbol{r}}\right) + \Omega^2\,\vec{\boldsymbol{r}}\end{aligned}$$

The *effective* acceleration $\vec{\boldsymbol{a}}_{rel}$ consists of 3 parts: the "true" acceleration $\vec{\boldsymbol{a}}$, the *centrifugal* acceleration $\Omega^2\,\vec{\boldsymbol{r}}$, and another term which we will discuss later. The centrifugal acceleration points in the direction of $\vec{\boldsymbol{r}}$ and is just (the opposite of) the acceleration of circular motion, as expected.

b) Cross Products

Introducing the angular velocity

$$\vec{\boldsymbol{\omega}} = \Omega\,\vec{\boldsymbol{k}}$$

it is easy to check directly that

$$\begin{aligned}\dot{\hat{\boldsymbol{r}}} &= \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{r}}\\ \dot{\hat{\boldsymbol{\phi}}} &= \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\phi}}\end{aligned}$$

[3] These are just the equations derived in the class notes for Paradigm 6, restricted to the special case of circular motion. Note however in what follows that $\vec{\boldsymbol{r}}$ has both an $\hat{\boldsymbol{r}}$ and a $\hat{\boldsymbol{\phi}}$ component, since we are using a basis adapted to the observer ($\vec{\boldsymbol{R}}$), rather than one adapted to the moving object.

But this means that for *any* relative vector of the form

$$\vec{\boldsymbol{F}}(t) = X(t)\,\hat{\boldsymbol{r}}(t) + Y(t)\,\hat{\boldsymbol{\phi}}(t)$$

we have

$$\begin{aligned}
\dot{\vec{\boldsymbol{F}}} &= \left(\dot{X}\,\hat{\boldsymbol{r}} + \dot{Y}\,\hat{\boldsymbol{\phi}}\right) + \left(X\,\dot{\hat{\boldsymbol{r}}} + Y\,\dot{\hat{\boldsymbol{\phi}}}\right) \\
&= \left(\dot{X}\,\hat{\boldsymbol{r}} + \dot{Y}\,\hat{\boldsymbol{\phi}}\right) + \left(X\,\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{r}} + Y\,\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\phi}}\right) \\
&= \left(\dot{X}\,\hat{\boldsymbol{r}} + \dot{Y}\,\hat{\boldsymbol{\phi}}\right) + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{F}}
\end{aligned}$$

Note that the first term is the "naive" derivative of $\vec{\boldsymbol{F}}$; this "naive" differentiation is precisely what was used to obtain $\vec{\boldsymbol{v}}_{rel}$ and then $\vec{\boldsymbol{a}}_{rel}$ starting from $\vec{\boldsymbol{r}}_{rel}$.

We can use $\vec{\boldsymbol{\omega}}$ to recompute "true" velocities and accelerations. Differentiating

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{R}} + \vec{\boldsymbol{r}}_{rel}$$

we obtain

$$\begin{aligned}
\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}} &= \dot{\vec{\boldsymbol{R}}} + \dot{\vec{\boldsymbol{r}}}_{rel} \\
&= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{R}} + (\vec{\boldsymbol{v}}_{rel} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{rel})
\end{aligned}$$

Further differentiation yields

$$\begin{aligned}
\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} &= \vec{\boldsymbol{\omega}} \times \dot{\vec{\boldsymbol{R}}} + \left(\dot{\vec{\boldsymbol{v}}}_{rel} + \vec{\boldsymbol{\omega}} \times \dot{\vec{\boldsymbol{r}}}_{rel}\right) \\
&= \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{R}}) + ((\vec{\boldsymbol{a}}_{rel} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{rel}) + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{v}}_{rel} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{rel})) \\
&= \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{R}}) + \vec{\boldsymbol{a}}_{rel} + 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{rel} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{rel})
\end{aligned}$$

Rewriting these expressions slightly, we obtain the following equations for the relative velocity and acceleration:

$$\begin{aligned}
\vec{\boldsymbol{v}}_{rel} &= \vec{\boldsymbol{v}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \\
\vec{\boldsymbol{a}}_{rel} &= \vec{\boldsymbol{a}} - 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{rel} - \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})
\end{aligned}$$

As before, the *effective* acceleration $\vec{\boldsymbol{a}}_{rel}$ therefore consists of 3 parts: the "true" acceleration $\vec{\boldsymbol{a}}$, the *centrifugal* acceleration $-\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$ and the *Coriolis* acceleration $-2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{rel}$. The centrifugal acceleration points in the direction of $\vec{\boldsymbol{r}}$, since

$$-\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) = \Omega^2 \vec{\boldsymbol{r}}$$

and is just (the opposite of) the acceleration of circular motion, as expected. Finally, for counterclockwise rotation ($\Omega > 0$), the Coriolis acceleration always points *to the right* of the direction of motion $\vec{\boldsymbol{v}}_{rel}$.

Note that the above expressions for $\vec{\boldsymbol{v}}_{rel}$ and $\vec{\boldsymbol{a}}_{rel}$ depend only on the angular velocity $\vec{\boldsymbol{\omega}}$, not on the particular choice of basis $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}\}$ nor the position of the rotating observer! Even though our derivation was basis-dependent, the result is therefore basis-independent, and the above expressions hold for *any* observer with angular velocity $\vec{\boldsymbol{\omega}}$, *including one at the origin.*