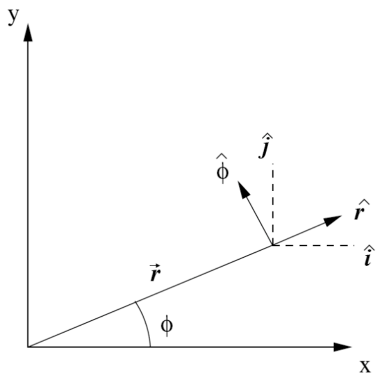
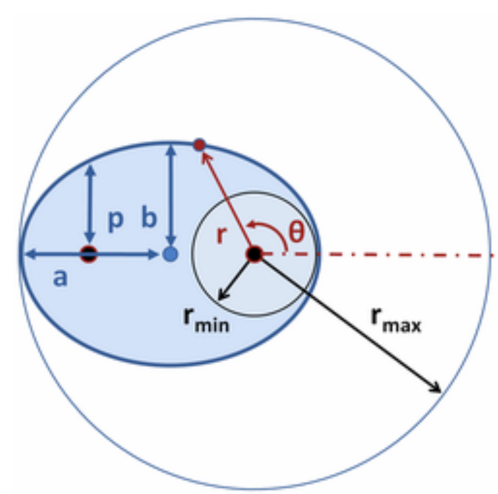


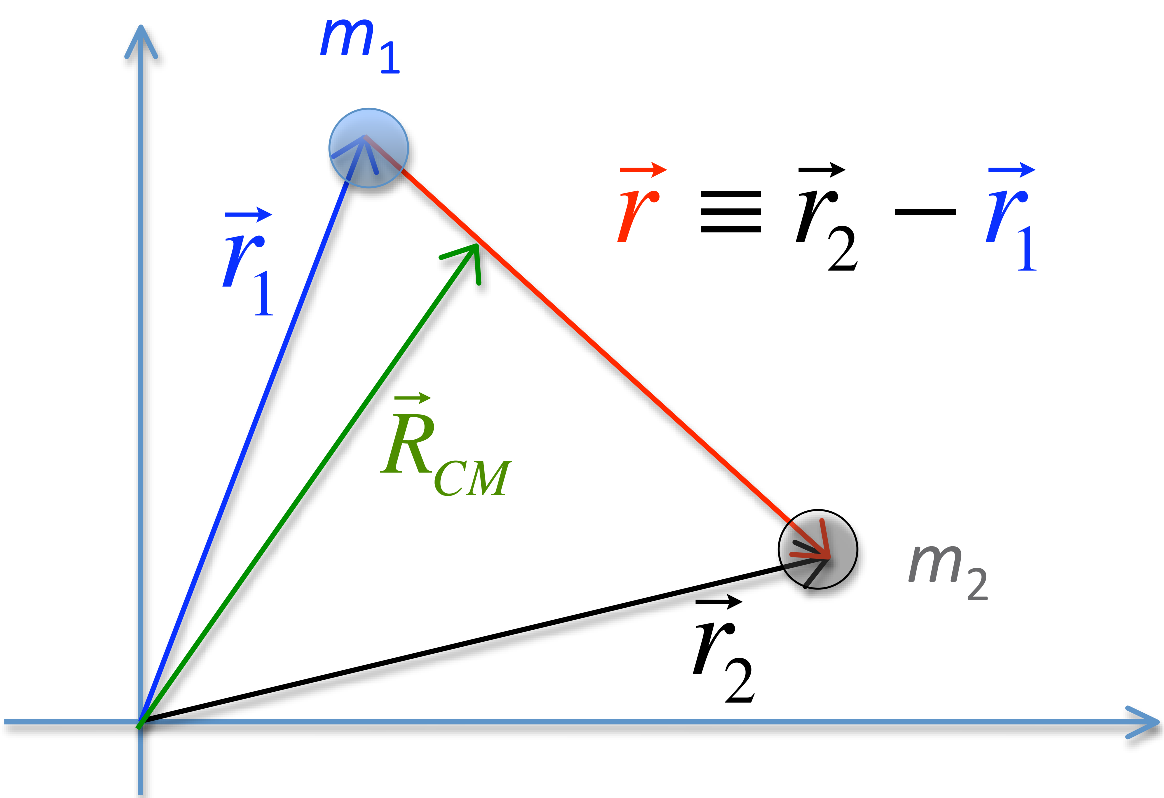
CM frame, relative coordinates, position and velocity in polar coordinates

Reading: Course packet



Relative coordinates useful

$$\vec{R}_{CM} \equiv \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2$$



$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1$$

Decouple 2-body problem

$$m_1 \ddot{\vec{r}}_1 = \vec{f}_{12} = -\vec{f}_{21}$$

$$m_2 \ddot{\vec{r}}_2 = \vec{f}_{21}$$

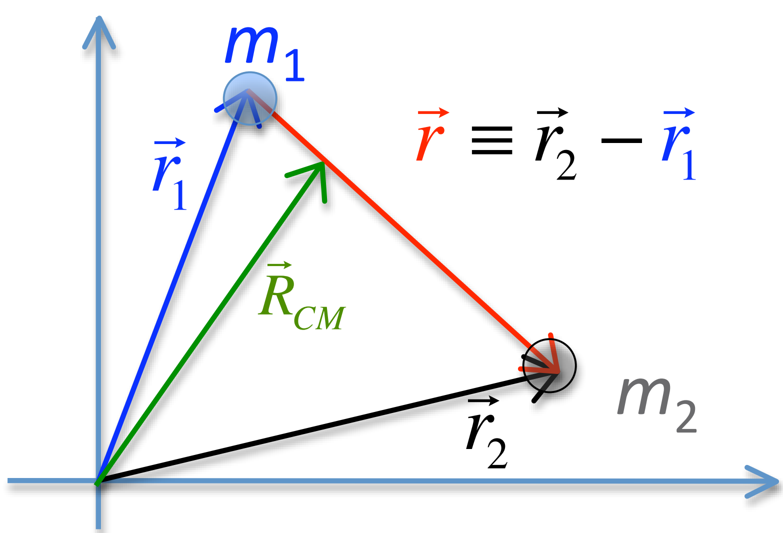
$$m_1 m_2 \ddot{\vec{r}}_2 - m_1 m_2 \ddot{\vec{r}}_1 = m_1 \vec{f}_{21} + m_2 \vec{f}_{21}$$

$$\frac{m_1 m_2}{m_1 + m_2} \ddot{\vec{r}} = \vec{f}_{21}$$

$$\mu \ddot{\vec{r}} = \vec{f}_{21} = f(r)\hat{r} \text{ for central force}$$

REDUCED MASS

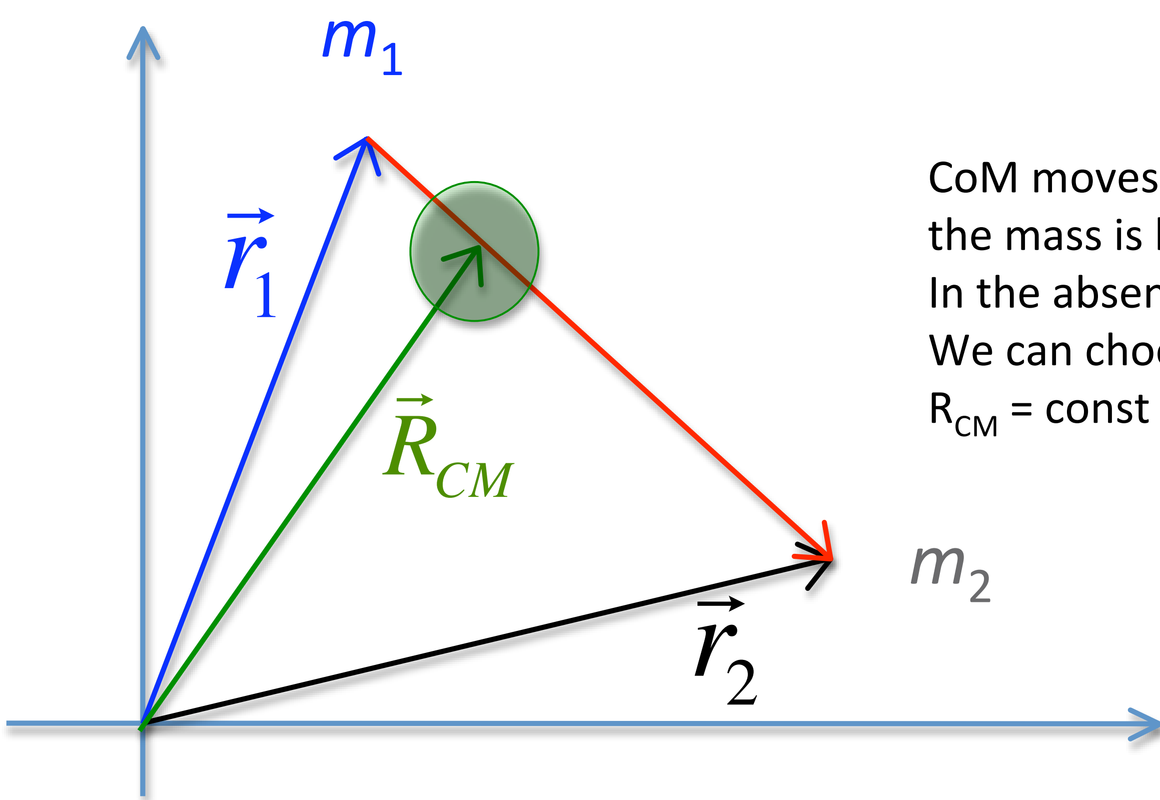
$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$



Relative motion obeys a single-particle equation. Fictitious particle, mass μ moving in a central force field. Vector r measures RELATIVE displacement; we have to "undo" to get actual motion of each of particles 1 and 2. Often m1>>m2.

Decoupled CoM motion: 2 body example

$$\vec{R}_{CM} \equiv \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2$$



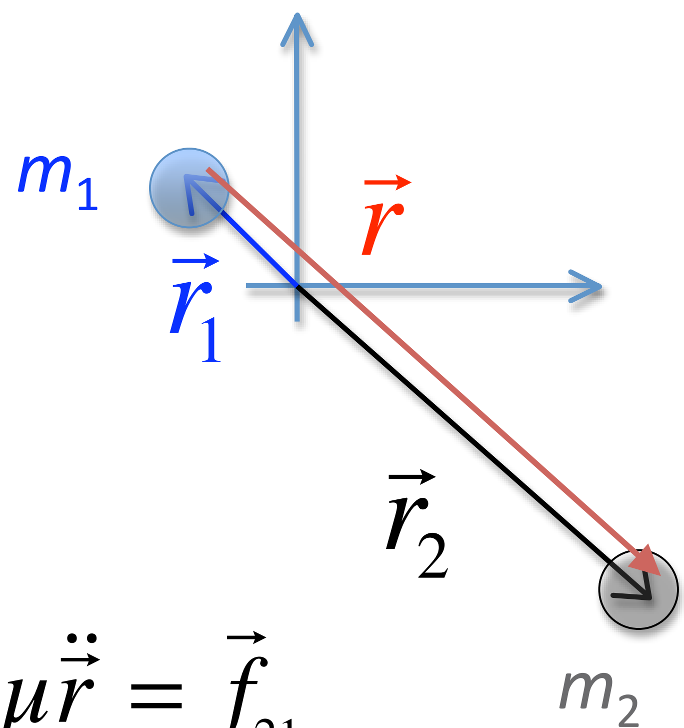
CoM moves subject to external forces as if all the mass is located there.
In the absence of ext forces, V_{CM} = constant
We can choose const vel = 0
R_{CM} = const and we can choose = 0.

CoM Frame

can choose if Fext=0

$$\vec{R}_{CM} \equiv \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2 \overbrace{=0}$$

$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1$$



$$\mu\ddot{\vec{r}} = \vec{f}_{21}$$

$$\vec{r}_2 = \frac{m_1}{m_1+m_2}\vec{r}; \qquad m_2\vec{r}_2 = \mu\vec{r}$$

$$\vec{r}_1 = -\frac{m_2}{m_1+m_2}\vec{r}; \qquad m_1\vec{r}_1 = -\mu\vec{r}$$

Relative motion obeys a single-particle equation. Fictitious particle, mass μ moving in a central force field. Vector r measures RELATIVE displacement; we have to “undo” to get actual motion of each of particles 1 and 2. Often m1>>m2.

Central force

- Define central force

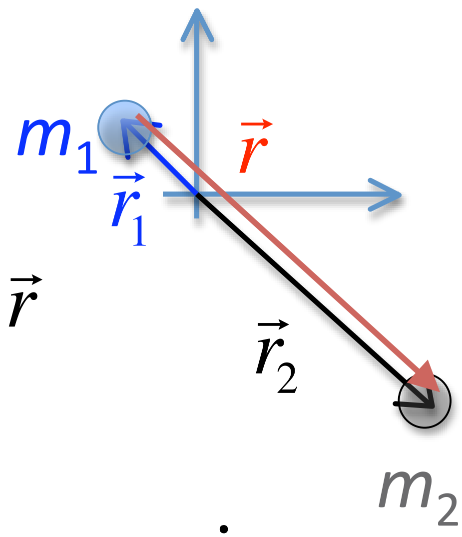
$$\vec{f}_{21} = -\vec{f}_{12} = f(r)\hat{r}; \quad \text{gravity: } f(r) = -\frac{Gm_1m_2}{r^2}$$

- f_{12} reads "the force on 1 caused by 2"
- Depends on magnitude of separation and *not* orientation
- Points towards origin in a 1-particle system
- Derivable from potential (conservative)

$$\vec{f}_{12} = -\nabla U(r); \quad \text{gravity: } U(r) = -\frac{Gm_1m_2}{r}$$

- Work done is path independent

2-body: L and T



$$m_1\vec{r}_1 = -\mu\vec{r} \;; \qquad m_2\vec{r}_2 = \mu\vec{r}$$

$$\vec{L}_{TOT} = \vec{r}_1 \times m_1\dot{\vec{r}}_1 + \vec{r}_2 \times m_2\dot{\vec{r}}_2 = \vec{r}_1 \times \left(-\mu\dot{\vec{r}}\right) + \vec{r}_2 \times \mu\dot{\vec{r}}$$

$$= \left(\vec{r}_2 - \vec{r}_1\right) \times \mu\dot{\vec{r}} = \vec{r} \times \mu\dot{\vec{r}}$$

$$T_{TOT} = \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} = \tfrac{1}{2} m_1 \frac{\mu^2}{m_1^2} \dot{\vec{r}}^{\,2} + \tfrac{1}{2} m_2 \frac{\mu^2}{m_2^2} \dot{\vec{r}}^{\,2}$$

$$= \tfrac{1}{2} \mu^2 \dot{\vec{r}}^{\,2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \tfrac{1}{2} \mu \dot{\vec{r}}^{\,2}$$

Conservation of angular momentum

- Define (must specify an origin!)

$$\vec{L} \equiv \vec{r} \times \vec{p}; \quad \vec{\tau} \equiv \vec{r} \times \vec{F}$$

- Newton's 2nd Law

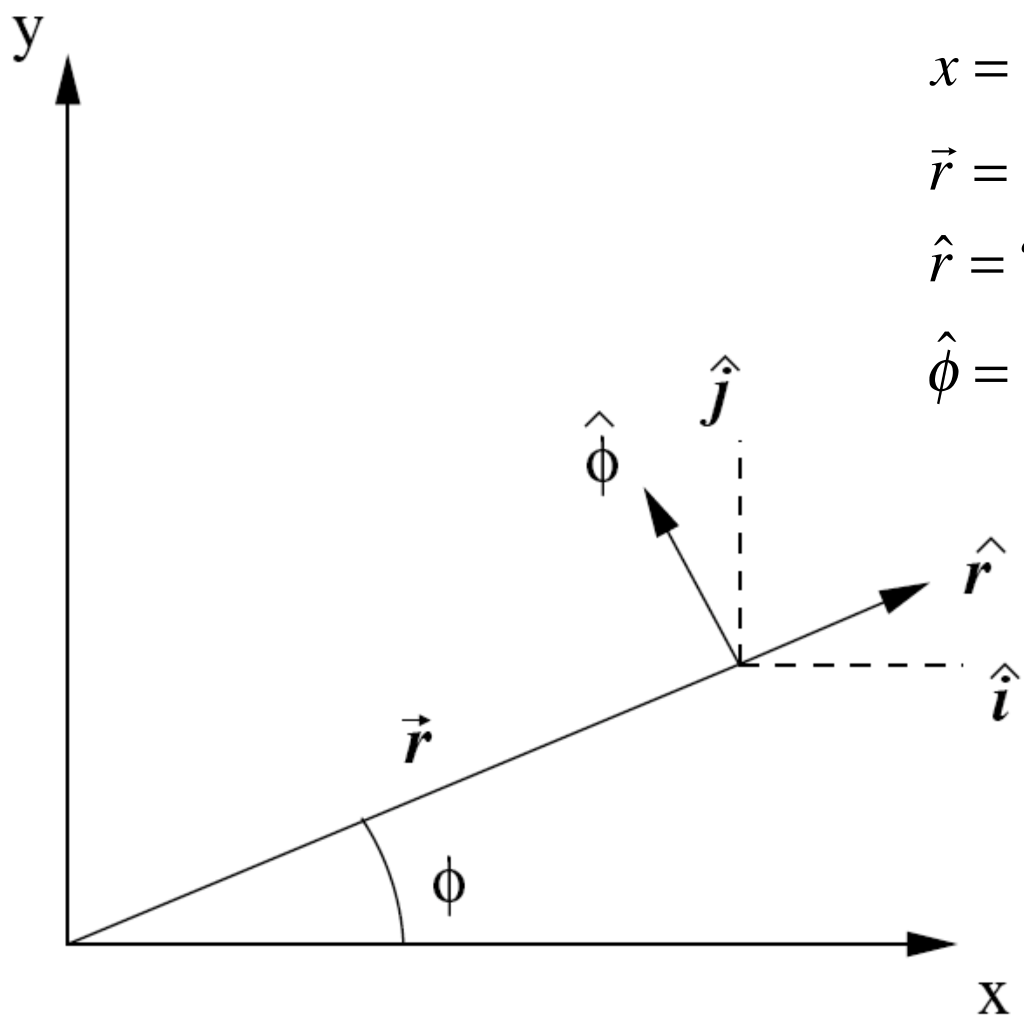
$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

- Central Force

$$f(r)\hat{r}$$

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = 0 \quad \Rightarrow \quad \frac{d\vec{L}}{dt} = 0$$

Polar coordinates are good for planar orbits



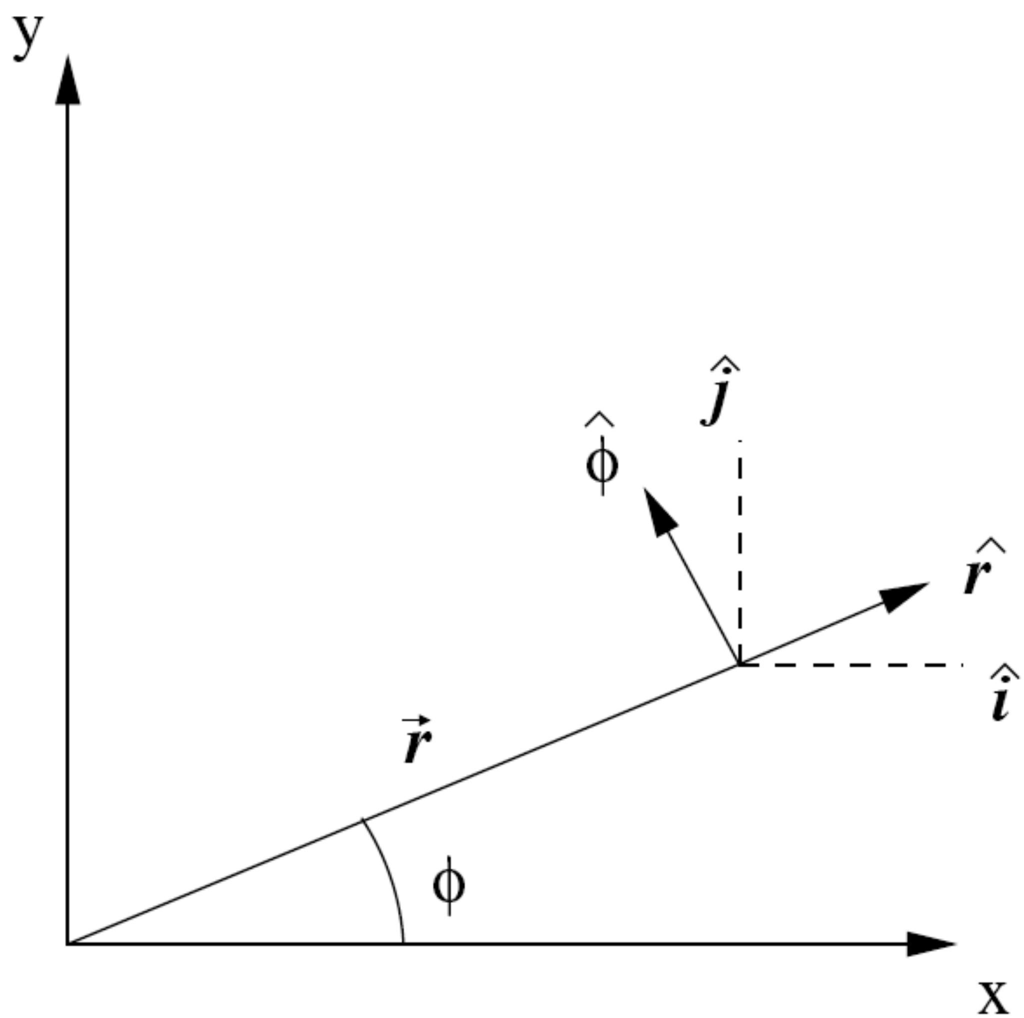
$$x = r\cos\phi; \quad y = r\sin\phi$$

$$\vec{r} = ?\,\hat{x} + ?\,\hat{y} = ?\,\hat{i} + ?\,\hat{j}$$

$$\hat{r} = ?$$

$$\hat{\phi} = ? \qquad \left[btw \quad \hat{x} \equiv \hat{i}; \hat{y} \equiv \hat{j} \right]$$

Polar coordinates are good for planar orbits

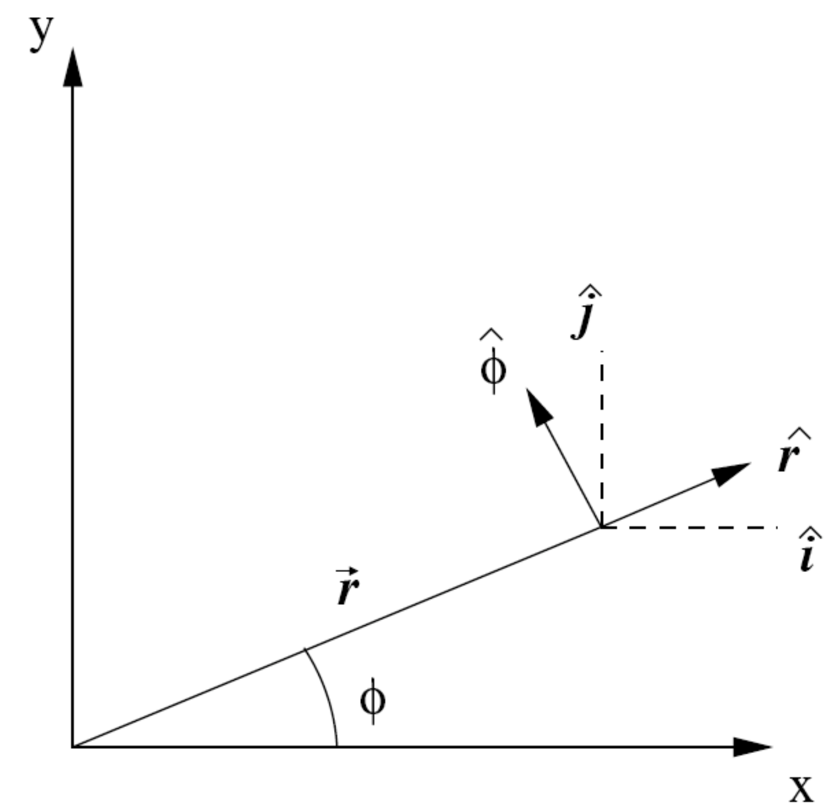


$$\frac{d\vec{r}}{dt} \equiv \dot{\vec{r}} = ?$$

$$\dot{\hat{r}} = ?$$

$$\dot{\hat{\phi}} = ?$$

Velocity and acceleration in polar coordinates



$$\dot{\hat{r}} = \dot{\phi}\hat{\phi}$$

$$\dot{\hat{\phi}} = -\dot{\phi}\hat{r}$$

$$\frac{d\vec{r}}{dt} \equiv \dot{\vec{r}} = \vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

$$\frac{d^2\vec{r}}{dt^2} \equiv \ddot{\vec{r}} = \vec{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

Angular momentum for the reduced mass problem

$$\vec{\ell} = \vec{r} \times \vec{p} = \vec{r} \times \mu\vec{v} = \vec{r} \times \mu\dot{\vec{r}}$$

$$= r\hat{r} \times \mu\left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\right)$$

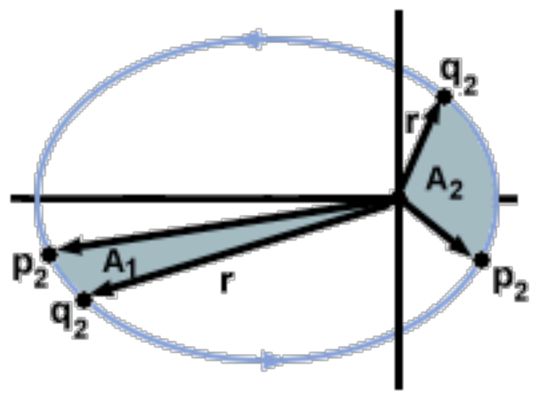
$$= -\mu r^2\dot{\phi}\hat{\theta} = \mu r^2\dot{\phi}\hat{z} = \mu r^2\dot{\phi}\hat{k}$$

$$\ell = \left|\vec{\ell}\right| = \mu r^2\dot{\phi} = const$$

Notice that if r and phi-dot (aka omega) are known at one time, then el is known at that time and thus for all times. INITIAL CONDITIONS!

Conservation of AM & Kepler's 2nd law

- Equal areas equal times (cons. angular mom.)
- Later (once we've explored polar coordinates).



Summary

- Define R_{CM}; total mass M
- Define r, relative coord; $\mu = m_1 m_2 / M$
- No ext forces => CoM constant momentum
- No external torque => AM conserved
- AM conserved -> what is value? (see later)
- AM conserved -> Orbit in a plane (hwk)