Central Forces Homework 5

Due 03/14/12

PRACTICE:

1. (McIntyre 7.20)

   In each of the following sums, shift the index $n \to n+2$. Don't forget to shift the limits of the sum as well. Then write out all of the terms in the sum (if the sum has a finite number of terms) or the first five terms in the sum (if the sum has an infinite number of terms) and convince yourself that the two different expressions for each sum are the same:

   (a)
   $$\sum_{n=0}^{3} n$$

   (b)
   $$\sum_{n=1}^{5} e^{in\phi}$$

   (c)
   $$\sum_{n=0}^{\infty} a_n n(n-1)z^{n-2}$$

2. (McIntyre 7.24 and 7.25)

   In spherical coordinates, the square of the angular momentum vector L^2 and the z-component of the angular momentum vector L_z are given by:

   $$L^2 = \vec{L}\cdot\vec{L} = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

   $$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

   (a) Show explicitly that the spherical harmonics are eigenvectors of L^2. What are the eigenvalues?

   (b) Show explicitly that the spherical harmonics are eigenvectors of L_z. What are the eigenvalues?

REQUIRED:

3. (a) By hand, find the recurrence relation for a power series solution $H(\rho)$ of the equation:

$$\rho\frac{d^2H}{d\rho^2} + (2\ell + 2 - \rho)\frac{dH}{d\rho} + (\lambda - \ell - 1)H = 0$$

   where ℓ is a known positive integer, and λ is an unknown constant.

   (b) Suppose that you want a solution to (a) which is a polynomial of degree 4. Assume that $\ell = 2$. What does that tell you about the unknown constant λ?

   (c) Find the polynomial of degree 4 solution to the differential equation in part (a) assuming $\ell = 2$. Assume anything you need to about λ.

4. (McIntyre 7.28)

   Use your favorite tool (*e.g.* Maple, Mathematica, Matlab, pencil) to generate the Legendre polynomial expansion to the function $f(z) = \sin(\pi z)$. How many terms do you need to include in a partial sum to get a "good" approximation to $f(z)$ for $-1 < z < 1$? What do you mean by a "good" approximation? How about the interval $-2 < z < 2$? How good is your approximation? Discuss your answers. Without using the worksheet, answer the same set of questions for the function $g(z) = \sin(3\pi z)$

5. (McIntyre 7.29)

   Consider the following normalized state for the rigid rotor given by:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|1,-1\rangle + \frac{1}{\sqrt{3}}\,|1,0\rangle + \frac{i}{\sqrt{6}}\,|0,0\rangle$$

   (a) What is the probability that a measurement of L_z will yield $2\hbar$? $-\hbar$? $0\hbar$?

   (b) If you measured the z-component of angular momentum to be $-\hbar$, what would the state of the particle be immediately after the measurement is made? $0\hbar$?

   (c) What is the expectation value of L_z in this state?

   (d) What is the expectation value of L^2 in this state?

   (e) What is the expectation value of the energy in this state?

   (f) (Challenge:) What is the expectation value of L_y in this state? L_y in spherical coordinates is given by:

$$L_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

6. Make a table, similar to the one you made for a particle confined to a ring, showing the different representations of the physical quantities associated with the rigid rotor. Include information about the operators $\hat{H}$, $\hat{L}_z$, and $\hat{L}^2$.

Rigid Rotor/Particle on a Sphere

	Ket Representation	Wave Function Representation	Matrix Representation
Hamiltonian			
Eigenvalues of Hamiltonian			
Normalized Eigenstates of Hamiltonian			
Coefficient of the energy eigenstate with quantum numbers ℓ, m			
Probability of measuring $E_{\ell,m}$			

Rigid Rotor/Particle on a Sphere

	Ket Representation	**Wave Function Representation**	**Matrix Representation**
Operator for square of the angular momentum			
Eigenvalues of L^2			
Normalized Eigenstates of L^2			
Coefficient of the eigenstates of L^2 with quantum numbers ℓ, m			
Probability of measuring $\hbar^2\ell(\ell+1)$ for the square of the angular momentum			

Rigid Rotor/Particle on a Sphere

	Ket Representation	Wave Function Representation	Matrix Representation
Operator for z-component of angular momentum			
Eigenstates of L_z			
Normalized Eigenstates of L_z			
Coefficient of m^{th} eigenstates of L_z			
Probability of measuring $m\hbar$ for z -component of angular momentum			