Capstones in Physics: Electromagnetism

1. ELECTROSTATICS

1.1. Electric Field
1.2. Electric Potential
1.3. Work and energy in Electrostatics
1.4. Conductors

(References to) "Introduction to Electrodynamics" by David J. Griffiths
1.1 Electric Field

Coulomb’s law

Force on a test charge $Q$ due to a point charge $q$

\[
\vec{F} = \frac{1}{4\pi \varepsilon_0} \frac{qQ}{r^2} \hat{r}
\]

\[
\vec{r} = \vec{R} - \vec{r}'
\]

\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}
\]

Permitivity of free space

\[
\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots
\]

\[
= \frac{Q}{4\pi \varepsilon_0} \left( \frac{q_1}{r_1^2} \hat{r}_1 + \frac{q_2}{r_2^2} \hat{r}_2 + \cdots \right)
\]

\[
= \frac{Q}{4\pi \varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i^2} \hat{r}_i
\]

Electric field

\[
\vec{F} = Q\vec{E} \quad \Rightarrow \quad \vec{E} = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i^2} \hat{r}_i
\]

superposition principle

For continuous charge distribution:

\[
\vec{E} = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(r)}{r^2} \hat{r} \, dv
\]
Gauss’s law
The flux through any surface enclosing charge $Q_{enc}$ is $Q_{enc}/\varepsilon_0$.

For any closed surface, $\iiint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$

Divergence theorem: $\iiint_V \nabla \cdot \vec{E} \, dV = \int \left( \nabla \cdot \vec{E} \right) d\tau$

$Q_{enc}$ in terms of the charge density $\rho$: $Q_{enc} = \int_{V} \rho \, d\tau$

$\Rightarrow \int_{V} \left( \nabla \cdot \vec{E} \right) d\tau = \int_{V} \left( \frac{\rho}{\varepsilon_0} \right) d\tau \Rightarrow \nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$

The Curl of $\vec{E}$
Integral of $\vec{E}$ around a closed path is zero:
$\oint \vec{E} \cdot d\vec{l} = 0 \Rightarrow \nabla \times \vec{E} = 0$

Stokes’ theorem, $\oint_S \vec{E} \cdot d\vec{l} = \int_S \left( \nabla \times \vec{E} \right) \cdot d\vec{a}$

1.2 Electric potential
Because $\oint \vec{E} \cdot d\vec{l} = 0$, the line integral of $\vec{E}$ from $a$ to $b$ is the same for all paths.
Because the line integral is independent of path, we can define a function

\[ V(\vec{r}) \equiv -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l} \]  

(electric potential).

O is some standard reference point, thus \( V \) depends only on the point \( \vec{r} \).

The differential version: \( \vec{E} = -\nabla V \)

**Poisson’s equations and Laplace’s equations**

\[
\vec{E} = -\nabla V \\
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0} \text{ Poisson’s equation} \\
\nabla \times \vec{E} = 0
\]

In regions \( \rho = 0 \), \( \nabla^2 V = 0 \) : Laplace’s equation

**Potential of a localized charge distribution**

The potential of a point charge \( q \):

\[ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}, \]  

where \( r \) is the distance from the charge to \( \vec{r} \).

For a volume charge, \( V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{r} d\tau' \)
1.3 Work and energy in Electrostatics

ENERGY OF A CHARGE DISTRIBUTION

potential energy of distribution = energy required to assemble it

construct from pair potential energies \( U_{ij} = \frac{q_i q_j}{4\pi \varepsilon_0 r_{ij}} \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>bring charge 1</td>
</tr>
<tr>
<td>2.</td>
<td>bring charge 2</td>
</tr>
<tr>
<td>3.</td>
<td>bring charge 3</td>
</tr>
<tr>
<td>4, 5,...</td>
<td></td>
</tr>
</tbody>
</table>

total energy = \( U_{12} + U_{13} + U_{23} + \ldots \)

\[ U_{total} = \sum_{pairs} U_{ij} \]

compare

\[ U_{total} = \frac{1}{2} \sum_i \sum_{j \neq i} U_{ij} = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{q_i q_j}{4\pi \varepsilon_0 r_{ij}} = \frac{1}{2} \sum_i q_i \sum_{j \neq i} \frac{q_j}{4\pi \varepsilon_0 r_{ij}} = \frac{1}{2} \sum_i q_i V_i \]

\( \Rightarrow \) generalization to continuous distribution has factor \( \frac{1}{2} \):

\[ U_{total} = \frac{1}{2} \int d^3r \, \rho(r) V(r) = \frac{1}{2} \int d^3r \, \int d^3R \, \frac{\rho(r) \rho(R)}{4\pi \varepsilon_0 |r-R|} \]

note this form doesn't work for idealized distributions, including point charges: the energy integral diverges!
ENERGY OF AN ELECTRIC FIELD

potential energy of a charge distribution = energy required to assemble it
(note: can only compute if finite!) see "energy of a charge distribution"

counting argument →

\[ U_{\text{total}} = \frac{1}{2} \int \int d^3r \ p(r) \ V(r) \]

express \( p \) in terms of electric field \( E(r) \):

\[ p(r) = \varepsilon_0 \nabla \cdot E(r) \]

substitute:

\[ U_{\text{total}} = \frac{\varepsilon_0}{2} \int d^3r \ V(r) \ \nabla \cdot E(r) \]

vector identity: \( \nabla \cdot (aA) = (\nabla a) \cdot A + a (\nabla \cdot A) \)

substitute \( a \rightarrow V, \ A \rightarrow E \):

\[ U_{\text{total}} = \frac{\varepsilon_0}{2} \int d^3r \ \{ \nabla \cdot [V(r) E(r)] - E(r) \cdot \nabla V(r) \} \]

express \( \nabla V \) in terms of electric field \( E(r) \):

\[ E(r) = -\nabla V(r) \]

divergence theorem: \( \int_V d^3r \ \nabla \cdot A(r) = \int_S dS \ \cdot A(r) \)

substitute \( A \rightarrow VE \), apply to expression for \( U_{\text{total}} \):

\[ U_{\text{total}} = \frac{\varepsilon_0}{2} \int dS \ \cdot [V(r) E(r)] + \frac{\varepsilon_0}{2} \int d^3r \ [E(r) \cdot E(r)] \]

first term \( \rightarrow 0 \) when surface goes to \( \infty \):

area \( \sim R^2 \), but localized charges \( \Rightarrow E \sim 1/R^2, \ V \sim 1/R \)

conclude

\[ U_{\text{total}} = \int d^3r \ \frac{\varepsilon_0}{2} \ E(r)^2 \]

but surely

\[ U_{\text{total}} = \int d^3r \ \text{(energy density)} \]

energy density \( = \frac{1}{2} \varepsilon_0 E(r)^2 \)
1.4 Conductors

STATIC CONDUCTORS

Conductors have charges free to move.

In steady state, constant current ⇒ motion of charges
In static state, charges aren't moving ⇒ no forces ⇒

\[
E(r) = 0 \text{ everywhere inside static conductor}
\]

⇒ 
\[
\rho(r) = 0
\]

and
\[
V(r) = \text{constant}
\]

⇒ charge density only on surface, \( \sigma \) see "Surface charge"

Gauss' law ⇒ \( E_{\text{outside}} - E_{\text{inside}} = \sigma/\varepsilon_0 \) normal to surface

but \( E_{\text{inside}} = 0 \), so

\[
E_{\text{outside}} = \frac{\sigma}{\varepsilon_0} \hat{n} \quad \text{at surface of conductor}
\]

Pressure on a conductor due to electric field

consider surface charge on area \( dS \)

\[
Q = \sigma \, dS
\]

force due to other charges

\[
F = Q \, E_{\text{external}}
\]

But \( E_{\text{external}} = E_{\text{due to surface charge}} \) ⇒

\[
E_{\text{external}} = \frac{1}{2} E
\]

pressure \( p = F_{\text{normal}}/dS \) ⇒

\[
p = \sigma \, E_{\text{ext}} = \frac{1}{2} \sigma E
\]

pressure \( p = \frac{\sigma^2}{2\varepsilon_0} = \frac{\varepsilon_0}{2} E^2 \) on surface of conductor

alternately, \( F = -\frac{\delta E_{\text{potential}}}{\delta \text{position}} \) see "Electrostatic Forces"

\[
E_{\text{pot}} = \frac{\varepsilon}{2} E^2 \times \text{vol} \quad \text{move surface} \Rightarrow \delta\text{vol} = S \times \delta \text{position}
\]
CAPACITANCE

recall a conductor is an equipotential,

claim: potential $V$ on an isolated conductor $\sim$ its charge $Q$

proof: $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ is a linear relation, determines $V$

definition: capacitance $C \equiv \frac{Q}{V}$ for isolated conductor

2 conductors with equal, opposite charges are a capacitor.

definition: capacitance $C \equiv \frac{Q}{|V_2 - V_1|} = \frac{Q}{\Delta V}$ for pair of conductors

energy stored by charging $= \int_0^Q dQ \, V(Q) = \int_0^Q dQ \, \frac{Q}{C}$

energy stored $= \frac{Q^2}{2C} = \frac{1}{2} Q \, V = \frac{1}{2} C \, V^2$

definition: 2 large flat plates
area $S$, dielectric thickness $d$
assume charges $\pm Q$ on plates
this is free charge!
$\Rightarrow \sigma = \frac{Q}{S} \Rightarrow$
$E - E_{\text{cond}} = \frac{\sigma}{\varepsilon_0}$
$E_{\text{cond}} = 0 \Rightarrow E = \frac{\sigma}{\varepsilon_0}$

$\Delta V = -\int dL \cdot E = d \, E = \frac{d\sigma}{\varepsilon_0} \Rightarrow C = \frac{Q}{\Delta V} = \frac{\sigma \, S}{d \, \sigma / \varepsilon_0} = \frac{\varepsilon_0 \, S}{d}$
MIRROR IMAGES

Uniqueness theorem: The solution to Laplace’s equation in some volume is uniquely determined if $V$ is specified on the boundary surface $S$.

At a plane interface, the potential can be described by a trick:

simplest example:

point charge $Q$ at $(-D,0,0)$
conducting plane $x = 0$

need to find induced charge to make $E \perp$ to plane

Potential for $x<0$ ($V=0$ for $x>0$)

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0} \left[ \frac{1}{\sqrt{(x+D)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x-D)^2 + y^2 + z^2}} \right]$$

Field in midplane

$$E_x(x=0,y,z) = \frac{2}{4\pi\varepsilon_0(D^2+y^2+z^2)} \frac{Q}{D} \frac{D}{\sqrt{D^2+y^2+z^2}}$$

surface charge density $\sigma(y,z) = \varepsilon_0 E \cdot \hat{n} = \frac{-QD}{2\pi(D^2+y^2+z^2)^{3/2}}$

Field on right = 0, field on left = $E_Q + E_{\text{induced}}$

$$E_{\text{left}} = \frac{Q(R+D\hat{x})}{4\pi\varepsilon_0[(x+D)^2+y^2+z^2]^{3/2}} + \frac{-Q(R-D\hat{x})}{4\pi\varepsilon_0[(x-D)^2+y^2+z^2]^{3/2}}$$

Note total induced charge = $-Q$ (proof: Gauss' law applied to surface enclosing all charges, field $\sim 1/r^3$)

generally, superposition $\Rightarrow$ mirror-image charge distribution

More complicated: image in surface of dielectric insulator
Other geometry: can also find images for surfaces:

- spherical surface
- cylindrical surface