

# Exponential Growth, Biological Case Study

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**Note:** This module assumes that the reader has first worked through the *Spontaneous and Exponential Decay* module, and particularly the parts dealing with the exponential function. The present module focuses on exponential growth and on understanding the half-life of biological growth.

In the *Spontaneous and Exponential Decay* module we described the average number of decaying nuclei remaining at a time  $t$  with the exponential function:

$$N(t) = N(0) e^{-\lambda t}, \quad (13)$$

where  $\lambda$  is the decay rate. If we reverse the sign in the exponential in eq. (13) from negative to positive (essentially reversing the sign of  $\lambda$ ), we obtain a model in which the population grows exponentially in time:

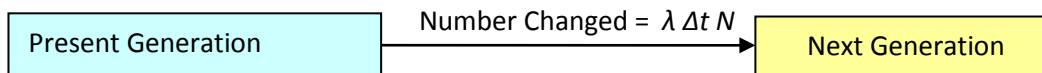
$$N(t) = N(0) e^{\lambda t}. \quad (1)$$

Although eq.(13) describes a population that decreases rapidly in time, while eq.(1) describes one that increases rapidly in time, the mathematics of the two are very similar. For example, in analogy with eq. (8) of exponential decay, exponential growth occurs when the average fraction of growth  $\Delta N/N$  in each time interval  $\Delta t$  is a constant:

$$\Delta N = \lambda \Delta t N. \quad (2)$$

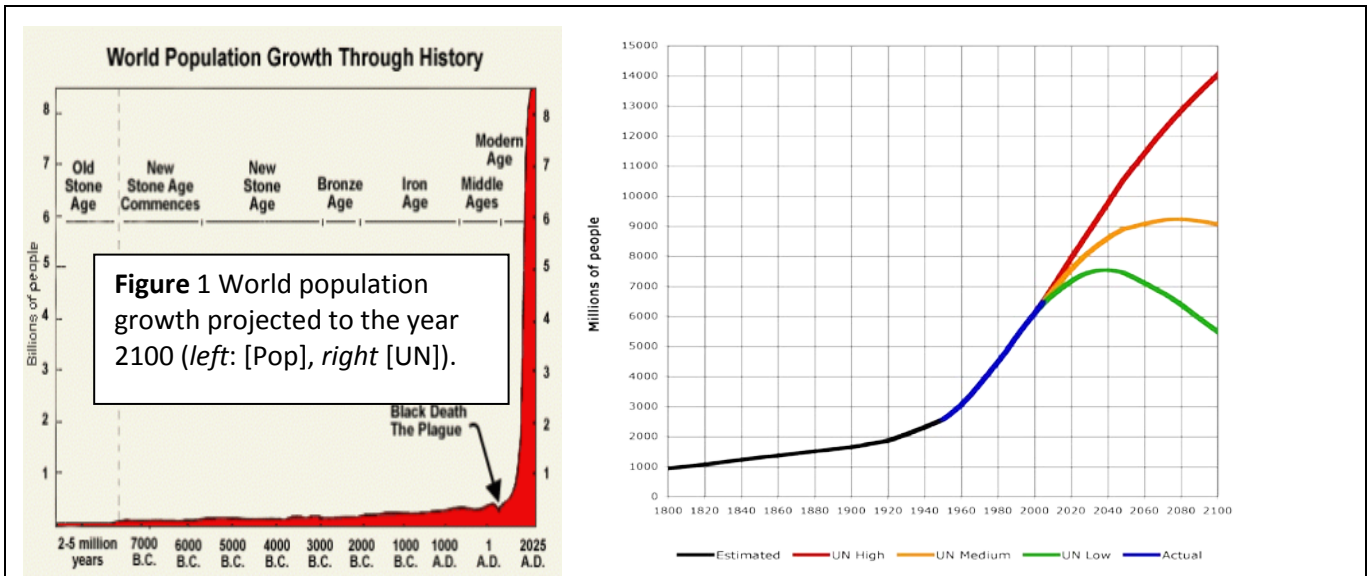
For example, eq. (2) would describe a situation where the number of births  $\Delta N$  in a population is proportional to the size  $N$  of that population, so as the population grows, so does the growth rate  $\frac{\Delta N}{\Delta t}$ .

As to be expected, a concept map describing exponential growth looks just like the one for exponential decay, except the number changed is now positive, indicating growth:

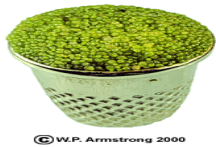


It is positive sign for the number changed here that leads to the positive sign in the exponential of eq. (1).

Exponential growth is a basic model for any reproducing population, to which refinements can be added. For instance, the growth of the microorganism *E. coli* in a test tube containing a *nutrient medium* appears to be exponential. In contrast, human population growth over the last 100 years is complicated, with some aspects of it predicted to be exponential (the projections in Figure 1) [Pop, UN].



In this module we consider *Wolffia*, the world’s smallest flowering plant. It is shaped like a microscopic green football, and is so small that 5,000 of them can fit into a thimble (Fig.2). An average individual plant is 0.6 mm long, 0.3 mm wide and weighs about 150 micrograms, approximately the weight of 2 grains of table salt. In addition to thimbles, *Wolffia* plant can be found floating at the surface of quite streams and ponds.



**Figure 2** A photograph of of *Wolffia* growing with a thimble [Wolf].

The growth time of *Wolffia* is measured in hours. A standard measure of the growth time is the *doubling time*  $\tau_2$ , that is, the time it takes a population to double. So if we start with an initial population of  $N(0)$  then at  $t = \tau_2$ ,

$$N(\tau_2) = 2 N(0) . \tag{3}$$

We can relate the doubling time  $\tau_2$  to the growth rate  $\lambda$  in Eq.(1) by evaluating Eq.(1) at  $t = \tau_2$ :

$$N(0) e^{\lambda \tau_2} = 2 N(0) . \tag{4}$$

If we take the natural logarithm of each side we obtain

$$\lambda \tau_2 = \ln 2 \rightarrow \tau_2 = \ln 2 / \lambda . \tag{5}$$

As might be expected, this is the same relation that the half life of a decaying nucleus has to its decay rate, Eq. (24) in the Spontaneous decay module.

- Why do you think the formulas for *half-life* and *doubling time* are the same?
- Can you show this mathematically?

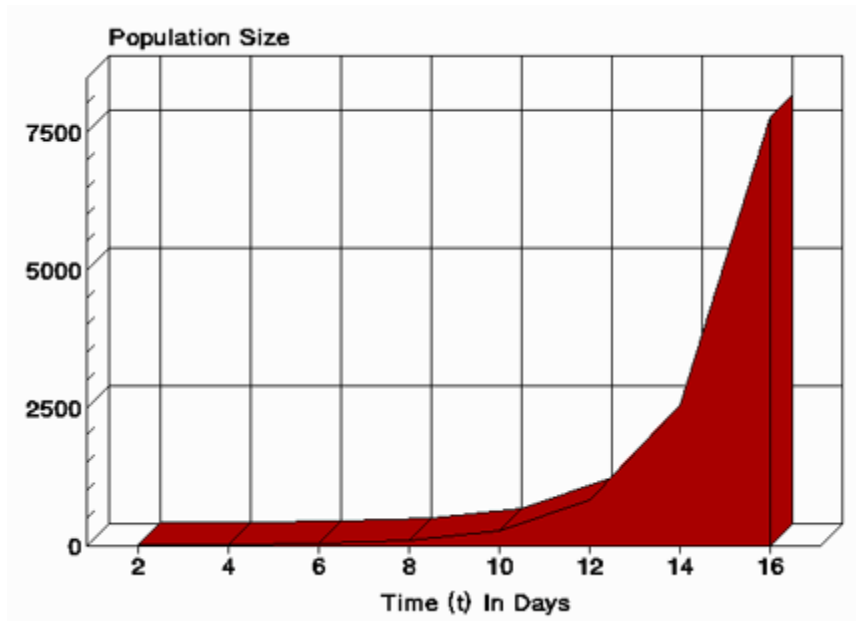
### Project

#### Objectives

- To determine the growth rate  $\lambda$  from the doubling time  $\tau_2$  for *Wolffia microscopica*. Although it is experimentally hard to measure the change in the number of plants over time, especially for those small guys in the thimble, it is straight forward to measure the change in weight or in size of the biomass over time.
- To reproduce a previously-measured growth as a function of time using our exponential model. (Reproducing someone else's results is often done in the early stages of modeling.)
- To examine the population growth curve and determine which portions, if any, are exponential.

### Steps

1. The doubling time for *Wolffia microscopica* via budding is 30 hours = 1.25 days. What is its growth rate  $\lambda$ ?
2. Reproduce and plot sixteen days of *Wolffia* growth.
3. Compare to the experimental results in Fig.3 using a time interval of  $\Delta t = 1$  day.
  - a. Use excel to generate a table of *Wolffia* population size vs. time for an initial population  $N(0) = 1$ .
  - b. Verify that the same results are obtained using either the doubling time or the growth rate.



### 4. Create Vensim simulation of *Wolffia* plant growth

- a. Save Vensim file of Radioactive Decay as **Wolffia growth** file
- b. Change the diagram to represent growth rather than decay
- c. Change the equations
- d. Use your Vensim model to reproduce the graphs you obtained in Excel.

### World Population Growth

On the right of Figure 1 is presented a United Nation's plot of the past growth of the world's population and three predictions of the future population.

1. As best you can, convert this graph into a table of numbers.
2. Determine which portions of this graph, if any, correspond to exponential growth or decay,  $N \propto e^{\pm k t}$ , where  $k$  is a constant. Examine all three predictions of the future.
3. Determine which portions of this graph, if any, correspond to linear decay or growth,  $N \propto m t$ , where  $m$  is a constant. Examine all three predictions of the future.
4. Determine which portions of this graph, if any, correspond to a power-law decay or growth,  $N \propto k t^n$ , where  $k$  and  $n$  are constants. Examine all three predictions of the future.

*Hint:* The hard way to do this is to try to fit different functions to the graph. The easy way to do this is to 1) make a semilog plot of the data [ $\log N(t)$  versus  $t$ ] and 2) a log-log plot of the data [ $\log N(t)$  versus  $\log t$ ]. Exponential behavior would appear as a straight line on a semilog plot, while power-law behavior would appear as a straight line on a log-log plot.

Proof:  $N = A e^{\pm k t} \rightarrow \log N = \pm k t + \log A$   
 $N = k t^n \rightarrow \log N = \log t + \log k$

## References

[Pop] *Eminent Scientist Claims Humans Will Be Extinct In 100 Years*, TheTechJournal, Technological News Portal, <http://thetechjournal.com/science/eminent-scientist-claims-humans-will-be-extinct-in-100-years.shtml> .

[UN] *World Population to 2300*, United Nations **Department** of Economic and Social Affairs, Population Division, <http://www.un.org/esa/population/publications/longrange2/WorldPop2300final.pdf>.

[Wolf] *Principles of Population Growth*, Wayne's Word, An On-Line Textbook of Natural History, <http://waynesword.palomar.edu/lmexer9.htm>.