Spin waves — excitations in a system of ordered spins. We will discuss SW in ferromagnets, using a semi-classical approach (which is a standard textbook approach). A "fully quantum" approach is also possible and often discussed by the same textbooks as a "more advanced step" - see, e.g. JMDC, pp. 164-65 — but due to our limited time we will not discuss it in detail.

A conceivable elementary excitation in an ordered spin system is one involving a "single spin flip":

Ground state: \( \ldots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \ldots \)

Excited state: \( \ldots \uparrow \uparrow \uparrow \downarrow \uparrow \uparrow \uparrow \ldots \)

\( S_i \)

\( Si \)

However, the energy of such an excitation is

\[ \Delta E = E(\downarrow) - E(\uparrow) = -2JS_i^\uparrow \cdot S_{i-1}^\uparrow - 2JS_{i+1}^\uparrow \cdot S_{i+1}^\uparrow - (2JS_i^\uparrow \cdot S_{i-1}^\uparrow - 2JS_i^\uparrow \cdot S_{i+1}^\uparrow) \]

and \( S_i^\uparrow S_{i-1}^\uparrow = S_i^\uparrow S_{i+1}^\uparrow = S^2 \); \( S_i^\uparrow S_{i-1}^- = S_i^- S_{i+1}^- = -S^2 \)

So \( \Delta E = 8JS^2 \). It might occur if \( k_B T \approx 8JS^2 \), and this appears to be way above the phase transition temperature! (see JMDC, p. 162).
So, the spin system "cannot afford" such "energy-expensive" excitations!

However, it should be kept in mind that the spin is actually angular momentum. Another type of excitation of an object possessing angular momentum is precession, and it appears to be much "less expensive" in the energy scale, and "affordable" for the spin system!

The classical description of precession is based on the 2nd Newton Law for rotational motion, viz.:

$$\frac{d\vec{L}}{dt} = \vec{T}$$

where \(\vec{T}\) is torque, and \(\vec{L}\) is the angular momentum.

The solution is an \(\vec{L}\) vector "moving on the surface of a cone":

One can write an analogous expression for a single spin, putting \(\hbar \vec{S}\) instead of \(\vec{L}\):
\[ \frac{d \vec{S}}{dt} = \gamma \vec{J} \]

Now, the torque: the torque acting on a classical magnetic dipole in an external field \( \vec{B} \) is simply

\[ \vec{\tau} = \vec{\mu} \times \vec{B} = \vec{\mu} \times (\mu_0 \vec{H}) . \]

Now, we can express \( \vec{\mu} \) of a single atom as \( \vec{\mu} = -g_\mu \vec{S} \).

And for \( \vec{H} \), in the spirit of the Mean Field Theory philosophy, we can put the "effective field" \( \vec{H}^{eff} \).

A standard textbook approach (GM, pp. 83-85; JHDC, p. 163) is to consider a 1-D chain of semi-classical spins:

\[ \ldots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \ldots \]

\[ \vec{S}_{j-1} \quad \vec{S}_j \quad \vec{S}_{j+1} \]

in which each spin \( \vec{S}_j \) interacts only with its nearest neighbours \( \vec{S}_{j-1} \) and \( \vec{S}_{j+1} \).

Then, the "effective field" \( \vec{H}^{ef} \) at the \( \vec{S}_j \) site is produced only by these two neighbours. As follows from the "practised notes" (Notes, pp. 68-73),

\[ \mu_0 H^{ef} = -\frac{2J(\vec{S}_{j-1} + \vec{S}_{j+1})}{g_\mu \alpha} \]

in such a case:
So, we get an equation:

\[
\hbar \frac{dS_j^z}{dt} = 2J \vec{S}_j \times (S_{j-1}^z + S_{j+1}^z)
\]

To solve, we expand the cross products in Cartesian coordinates:

\[
(\vec{S}_j \times \vec{S}_{j-1})_x = S_j^y S_{j-1}^z - S_j^z S_{j-1}^y
\]

And so on;

we get:

\[
\frac{dS_i^x}{dt} = \frac{2J}{\hbar} \left[ S_j^y (S_{j-1}^z + S_{j+1}^z) - S_j^z (S_{j-1}^y + S_{j+1}^y) \right]
\]

And similarly, for \( \frac{dS_i^y}{dt} \) and \( \frac{dS_i^z}{dt} \).

Now, there is an important step. We choose \( z \) as the axis of the "precession cone":

And we assume that the excitation is small, so that \( S_j^x, S_j^y \ll S \)

Then, \( S_j^z \approx S \).

In such an approximation, we can drop all \( S^x S^y \) terms, and leave only the \( S^x S^z \) and \( S^y S^z \) \( \approx S^y \) terms. Then, our 3-equation system simplifies to:
\[ \frac{dS^x_j}{dt} = \frac{2JS}{h} (2S^y_{j-1} - S^y_j - S^y_{j+1}) \]
\[ \frac{dS^y_j}{dt} = -\frac{2JS}{h} (2S^x_j - S^x_{j-1} - S^x_{j+1}) \]
\[ \frac{dS^z_j}{dt} = 0 \]

This equation system is similar to that one obtains for elastic waves in the theory of lattice vibrations (phonon theory). Have you done that theory in the earlier "block" of this "modular course"?

Well, as in the case of lattice vibration theory, we seek "solutions" in the form of propagating plane waves, which can be written as:

\[ S^x_j = u e^{i(k \cdot a \cdot j - \omega t)} \]  Here \( a \) is the "lattice constant", so that \( a \cdot j \)
\[ S^y_j = v e^{i(k \cdot a \cdot j - \omega t)} \]  is the distance along the chain

By inserting them into our equations, we get:
\[-i\omega u = \frac{2JS}{\hbar} \left(2 - e^{-ika} - e^{ika}\right) u = \frac{4JS}{\hbar} (1 - \cos ka) u\]
\[-i\omega v = -\frac{2JS}{\hbar} \left(2 - e^{-ika} - e^{ika}\right) u = -\frac{4JS}{\hbar} (1 - \cos ka) u\]

These equations have solutions for \(u\) and \(v\) if the denominator set up from the multipliers of \(u\) and \(v\):

\[
\begin{vmatrix}
  i\omega & \frac{4JS}{\hbar} (1 - \cos ka) \\
-\frac{4JS}{\hbar} (1 - \cos ka) & i\omega
\end{vmatrix} = 0
\]

From which we get:

\[
\hbar w = 4JS (1 - \cos ka)
\]

Putting that back into the equations at the top of the page, we obtain (from any of them) that \(v = -i\omega u\).

If we put the above back into the initial "test solutions", then we get their \textit{real} parts as:

\[
S_x^* = u \cos (ka - \omega t); \quad S_y^* = v \sin (ka - \omega t)
\]
whis is indeed a precession motion solution for the $\vec{S}_j$ vector.

Furthermore, if we write these solutions for several successive spins, $\vec{S}_j, \vec{S}_{j+1}, \vec{S}_{j+2}...$

\[
\begin{align*}
S_j^x &= u \cos(kaj - \omega t) \\
S_j^y &= u \sin(kaj - \omega t) \\
S_{j+1}^x &= u \cos(kaj - \omega t + ka) \\
S_{j+1}^y &= u \sin(kaj - \omega t + ka) \\
S_{j+2}^x &= u \cos(kaj - \omega t + 2ka) \\
S_{j+2}^y &= u \sin(kaj - \omega t + 2ka)
\end{align*}
\]

As follows from the above, there is a phase shift of $ka$ between the precession of any $\vec{S}_j, \vec{S}_{j+1}$ spins. The effect can be illustrated by the pictures below, showing the "side view" and the "top view":

\[\text{a) \quad \text{b)}\]

In other words, there is "precession wave" propagating along the chain, with the phase shift changing periodically, with a period $\lambda = \frac{2\pi}{ka}$. $\lambda = \frac{2\pi}{k}$
The equation we got earlier, \( \hbar \omega = 4JS(1-\cos ka) \), is the dispersion relation for the spin waves.

For small values of \( ka \), \( \cos ka \approx 1 - \frac{(ka)^2}{2} \)

so that

\[
\hbar \omega \approx 2JSa^2 \cdot k^2 = D_{sw} \cdot k^2
\]

It means that at the beginning the dispersion relation is quadratic.

\( \hbar \omega \), when using the quantum-mechanical treatment of the problem, can be interpreted as the energy of a SW quantum, commonly referred to as "magnon."

There is an analogy with elastic waves in crystal, in which case a quantum is called a "phonon". However, in contrast to magnons, the dispersion relation for phonons in the small-\( k \) region (i.e., for sound waves) is not quadratic, but linear.

\( D_{sw} = 2JSa^2 \) is called "the spin wave stiffness constant"
In the quantum dynamics, to find from a general theory of propagating excitations in periodic lattices, the maximum value of $k$ is $k = \frac{\pi}{a}$, which corresponds to the Brillouin zone boundary. So, the dispersion relation for the entire range of allowed $k$ values look like this: