Chapter 1

Oscillations and Waves – a Simple-Minded General Picture

1.1 Memory Refresher: The Derivative of a Continuous Function

I hope you all know what the function derivative is – so it's only for refreshing your memory. Suppose that there is a function, $y = f(x)$, that can be presented graphically as a continuous curve in a rectangular XY coordinate system – as in the Fig. 1.

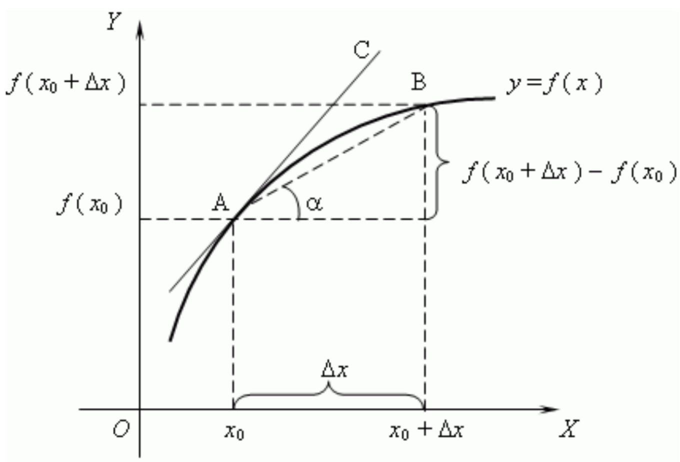

Figure 1.1: A curve representing an $y = f(x)$ function.

Let A be a point on the curve with x, y coordinates $x = x_0$ and $y = y_0 = f(x_0)$. We want to find the slope of the line tangent to the curve and passing through the A point. How it can be done? Well, let's select another point on the curve, call it B, some distance Δx to the right from A – so that its coordinates are $x = x_0 + \Delta x$ and $y = f(x_0 + \Delta x)$. Now, lrt's connect the A and B points with a line. It makes an angle α with the horizontal direction. Looking at the Fig. 1, it is easy to see that the tangent of α angle can be written as:

$$\tan\alpha = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{1.1}$$

You may start protesting: *But the line AB is not the* ***tangent*** *line we are supposed to find!* – you are absolutely right. But what we have done was only the beginning (because the course will be on music, we can say: *the overture*). And we shall continue: let's move the B point gradually, step by step, closer and closer to point B, so that Δx gets smaller and smaller. The steps have to be made smaller and smaller, to be sure the B point is always to the right of A. With each step, the AB line becomes closer and closer to the tangent line passing through A. So, in the $\Delta x \to 0$ limit, the slope of the AB line identical with the slope of the "real" tangent line in question.

$$\tan\alpha_{\text{tangent}} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{1.2}$$

The expression on the right side of the above equation is called the *derivative* of the $f(x)$ function. There are several equivalent symbols of a derivative, the two most commonly used "styles" are:

$$\text{One is} \quad f'(x)\ ; \quad \text{and the other is :} \ \ \frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx}f(x)$$

There is yet another symbol of a derivative, introduced by Newton – a dot above the character symbolizing the function, e.g., $\dot{x}$ – but it restricted only for **functions of time**, such as the distance traveled by a moving object: $x = x(t)$, or the object's velocity: $v = v(t)$. But the derivatives can also be written in the way shown above, i.e.:

$$\dot{x} \equiv \frac{dx}{dt} \quad \text{or} \quad \dot{v} \equiv \frac{dv}{dt}$$

where the $\equiv$ symbol means: *is equivalent to.*

In conclusion: for a given continuous function $f(x)$, for every value x_0 of the independent variable x, there exists a *derivative* defines as:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{1.3}$$

The slope of the line tangent to the curve representing the $f(x)$ function, passing through the point $f(x_0)$ on the curve, is equal to the derivative $f(x_0)$:

$$[\tan\alpha]_{x=x_0} = f'(x_0)$$

If you prefer the $df(x)/dx$ notation, you may write instead:

$$[\tan\alpha]_{x=x_0} = \left[\frac{f(x)}{dx}\right]_{x=x_0}$$

Sometimes, for convenience, we drop the (x) and instead of $df(x)/dx$ we write simply df/dx – e.g., if $f(x)$ is the only function we discuss at a given moment.

One common not-100%-true statement people often make: *the derivative is the slope of a line tangent to the curve representing a $f(x)$ function.* No, no! The **principal and the most general definition of a derivative** is that given by the Eq. 1.3. As far as the slope is concerned, it is OK to say: *the slope of a line tangent to the curve representing a $f(x)$ function is equal to the derivative of this function.* The same? No, not exactly. There may exist functions that do have derivatives, even though they cannot be plotted in the form of a simple curve.

The derivative $y' = f'(x)$ of a function $Y = f(x)$ is also a function! Really? Yes. What does it mean that y is a function of x, i.e., $y = f(x)$? It means that that to each arbitrarily chosen value of the independent value x – call it x_0 – there is attributed a value of the dependent variable y. Only a **single value** y_0, remember! – i.e., $y_0 = f(x_0)$ (if there are two or more y values attributed to the same value of x, it's not a function! Therefore, a full circle cannot be though of as a curve representing a function; only a semi-circle can be).

The full name of the derivative defined by the Equation 1.3 is "the **first** derivative". OK, then one may expect that there is also a **second derivative**. Absolutely correct! How comes?

Well, if attributing a single value of $y_0 = f(x_0)$ to any arbitrarily chosen value of x_0 defines a function, then – as logically follows from the Eq. 1.3, the derivative is also a function! Because a single value of $f'(x_0)$ is attributed to each x_0 value!

So, if $f'(x)$ is a function, it also should have a derivative, right? Right! We can denote tis derivative as $(f'(x))'$, or, for simplicity, as $f''(x)$ (which reads as: *ef* ***double prime*** *of ex*. The mathematical definition is then:

$$f''(x_0) = \lim_{\Delta x \to 0} \frac{f'(x_0 + \Delta x) - f'(x_0)}{\Delta x} \tag{1.4}$$

Combining the Eq. 1.4 with Eq. 1.3, the above definition can be written in a more elaborate form:

$$f''(x_0) = \lim_{\Delta x \to 0} \frac{\lim_{\Delta x \to 0} \frac{f(x_0 + 2\Delta x) - f'(x_0 + \Delta x)}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}{\Delta x} \tag{1.5}$$

Which can be rewritten in a simpler form, as a single limit expression:

$$f''(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + 2\Delta x) + f(x_0) - 2f(x_0 + \Delta x)}{(\Delta x)^2} \tag{1.6}$$

Another symbol for a second derivative, equivalent to $f''(x)$, is:

$$f''(x) \equiv \frac{d^2 f(x)}{dx^2} \equiv \frac{d^2}{dx^2} f(x)$$

In the same way one can define the third derivative, the fourth derivative, and so on. The respecive symbols will be:

$$f'''(x) \equiv \frac{d^3 f(x)}{dx^3} \qquad \text{and} \qquad f''''(x) \equiv \frac{d^4 f(x)}{dx^4}.$$

For such "high-order derivatives" the $d^n f(x)/dx^n$ are more often used because "too many primes" at after f become confusing.

Calculating derivatives using the basic Eq. 1.3 may be a good exercise. Take, for instance, the function $y = x^2$. From the equation, we find:

$$y' = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

But Ph331 is not a math course, so we will not do more examples over here.

1.2 The Motion of a Simple Oscillator

We will begin with presenting something that is usually referred to as a *Simple Harmonic Oscillator* (SHO), or, more specifically, a *Simple Harmonic Spring Oscillator*. It's a very simple device, it consists of only of a spring and a weight of mass m. On the Web, one can find a whole bunch of pictures and animations showing a spring oscillator – here is one example on YouTube, but you may find many more.

If one stretches a spring, it "protests" – it does not want to be distorted. The "protest" manifests itself as a "pulling-back" force. Suppose that the end of a non-stretched spring is at $y_0 = 0$. Now, you pull it, it elongates, so the end is not at a new position Δy. The spring "protests", and creates a force F that tends to restore its original unstretched length. Therefore, we call this force a *restoring force*. There is a simple relation between the elongation Δy and the restoring force F, known as the Hooke's Law. It states that F is proportional to Δy:

$$\boxed{F = -k \cdot \Delta y \quad (1.7)}$$

where k is a coefficient called the *spring constant*, or the ***spring's stiffness constant***. Its value characterizes a given spring, and it has to be individually determined for each spring – which is not difficult. For example, one may suspend a mass m from the spring and measure how much the spring extends. The weight pulls down the spring with a force of $m \cdot g$ (where $g = 9.81$ m/s^2 is the acceleration due to gravity), and the spring tends to pull the weight up with a force of the same magnitude. Let's "downwards" be the "positive direction". Hence, the restoring force in the present case is $F = -m \cdot g$, and from the Eq. 1.7 we get:

$$\boxed{k = -\frac{F}{\Delta y} = -\frac{-m \cdot g}{\Delta y} = \frac{m \cdot g}{\Delta y}. \qquad (1.8)}$$

The spring extension Δy (we can also call it the *displacement* of the weight m) can be readily measured using a ruler, and after carrying out the multiplication and the division, we get the value of k. From the Eq. 1.8 it follows that the unit of k must be a (unit of force)/(unit of length), i.e., N/m. The unit of the expression in the numerator is (mass uni)·(acceleration unit), i.e., kg·m/s^2 which is the same the Newton, N, and the unit of Δy is a meter, m, so we indeed get k with the right unit.

One more comment: why do we put minus in the Eq. 1.8? Well, the answer is simple, F, the restoring force, always "pulls" in opposite direction relative to the extension, and therefore the minus sign. Now, lets consider

the motion of such an oscillator. In order to make things even simple, let's consider not a weight suspended from a vertical spring, but an oscillator lying horizontally on a table. The other end of the spring is at a fixed position. In such way the only force acting on the mass m is the spring's restoring force, and we don't need to include the gravity – with a single force, the problem indeed gets simpler. You may protest, however: *Simpler? And what about the friction?!*. No, there are nearly frictionless tables – e.g., utilizing an air-cushion – so we can assume that we are using one of them, and we don't need to worry about friction.

So, the only force acting on the weight at a position other than than the equilibrium point x_0 is the spring's restoring force. According to the Second Law of Dynamics, a body of mass m acted upon by a force F moves in a direction parallel to the force vector, with acceleration:

$$a = \frac{F}{m} \tag{1.9}$$

Now, we need to start using some calculus elements.

*** Students allergic to calculus, you may skip the calculations, an important ***
**** thing is only that you understand the final result given by the Eq. 1.12 ****

The acceleration is the second time derivative of the displacement Δy:

$$a = \frac{d^2\Delta y}{dt^2} \tag{1.10}$$

By combining the Eqs. 1.8, 1.9 and we obtain:

$$\frac{d^2\Delta y}{dt^2} = -\frac{k}{m}\Delta y \tag{1.11}$$

What we got is the so-called *equation of motion* of the weight m. It's a second-order differential equation (most equations of motion in physics are). The solutions of differential equations are not numbers (like in the case of algebraic equations), but *functions*. We will not explain here how to solve such equations – we will simply show you the solution:

$$\Delta y(t) = A\sin\left(\sqrt{\frac{k}{m}}\cdot t\right) \tag{1.12}$$

We write $\Delta y(t)$ to stress that the displacement is a function of time. Clearly, it is an oscillating function. The A coefficient is called the *amplitude*, and it's the maximum displacement to the left and to the right that may occur in the course of the oscillation process.

I (Dr. Tom) dont like to pass information to students by telling them: *It is so!* and then exclaiming: *You have to believe me!*. So, we have to show that the Eq. 1.12 is indeed a solution of the equation of motion. How we do that? It's simple, we plug the solution into the equation and check if the left side (L) indeed is equal to the right side (R). So, at the left side we have the second derivative, which means that we have to differentiate the function twice. We have to use the general differentiation formulas for $\sin(C\cdot t)$ and $\cos(C\cdot t)$ functions, where B and C are arbitrary constants:

$$\frac{d}{dt}B\sin(C\cdot t) = BC\cdot\cos(C\cdot t), \quad \text{and} \quad \frac{d}{dt}B\cos(C\cdot t) = -BC\cdot\sin(C\cdot t)$$

OK, so after the first differentiation of the solution function we get:

$$\frac{d}{dt}A\sin\left(\sqrt{\frac{k}{m}}\cdot t\right) = A\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\cdot t\right)$$

And after the second:

$$\frac{d}{dt}A\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\cdot t\right) = -A\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}\cdot t\right)$$

And note that this is exactly the same what we get after inserting the expression for Δy in the Eq. 1.12 into the right side of the Eq. 1.11. So, indeed we get L = R, and Eq. 1.12 is a correct solution of the equation of motion.

** Who decided to skip the calculus-infested part, may continue from here **

Now, we only have to explain what is the physical meaning of the $\sqrt{k/m}$ expression. Suppose that at certain arbitrarily chosen instant of time t' the displacement is $\Delta x(t')$. Suppose that T is the time period needed for making one full oscillation cycle. Therefore, the displacement at t' is the same as at $t' + T$:

$$\Delta x(t') = \Delta x(t' + T)$$

In other words, it must be:

$$A\sin[\sqrt{\frac{k}{m}}\cdot t'] = A\sin[\sqrt{\frac{k}{m}}\cdot (t' + T)] \tag{1.13}$$

And we know that the properties of the $\sin(\alpha)$ function are such for any α:

$$\sin(\alpha) = \sin(\alpha + 2\pi)$$

In view of the above, it means th at the relation between the arguments in the Eq. 1.13 is

$$\sqrt{\frac{k}{m}}\cdot (t' + T) = \sqrt{\frac{k}{m}}\cdot t' + 2\pi$$

From that we get:

$$\sqrt{\frac{k}{m}}\cdot T = 2\pi$$

meaning that ***the oscillation period of the oscillator is:***

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{1.14}$$

Most oscillations we deal with in real life are fast, and very fast. Therefore, the oscillation periods are very small numbers – not too convenient to work with. A much more "user-friendly" parameter is the ***frequency*** f, i.e., the number of oscillation cycles per one second:

$$\boxed{f = \frac{1}{T} \qquad (1.15)}$$

By combining the Eqs. 1.14 and 1.15, we get:

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (1.16)}$$

But physicists are not fully satisfied with the frequency f, they prefer something else they call the ***angular frequency***, which is denoted as ω (lower case Greek omega) and is simply the "ordinary" frequency multiplied by 2π: $\omega = 2\pi f$. Then:

$$\boxed{\omega = \sqrt{\frac{k}{m}} \qquad (1.17)}$$

and the Eq. 1.12, describing the time dependence $\Delta y(t)$ of the oscillator's displacement, takes a simple and compact form:

$$\boxed{\Delta y(t) = A \sin(\omega \cdot t) \qquad (1.18)}$$

However, this in not the **only** solution of the oscillator's equation of motion. It can be readily checked that if you add an arbitrary constant number Φ to the sine function argument $\omega \cdot t$, it will also be a solution to the oscillator's equation of motion given by the Eq. 1.11. So, in general, the function describing the motion of the oscillator can be written as:

$$\boxed{\Delta y(t) = A \sin(\omega \cdot t + \Phi) \qquad (1.19)}$$

The Φ number is commonly referred to as the ***phase angle***. The motion of two identical oscillators may not be identical, if their Φ angles are different – we say then that there is ***phase shift between them***. A good illustration of a phase shift may be a simple experiment with two identical pendulums, which will be shown in class (even though a simple pendulum, i.e., a bob suspended on a piece of string, is not a *spring oscillator*, it **is** an oscillator, and it may be very helpful for explaining certain characteristics of oscillatory motion).

it's not

1.3 The Equation of Wave Motion

We are supposed to talk about waves – so why have we used several pages to talk about oscillators, which obviously are ***not*** waves?

The answer is simple. Once mathematical description of simple oscillators is understood, it is easier to understand the equations describing waves! Reason Two, more important: a simple oscillator is a system the motion of which is periodic *in time*. And a wave motion is a motion that is periodic *both in time and space*.

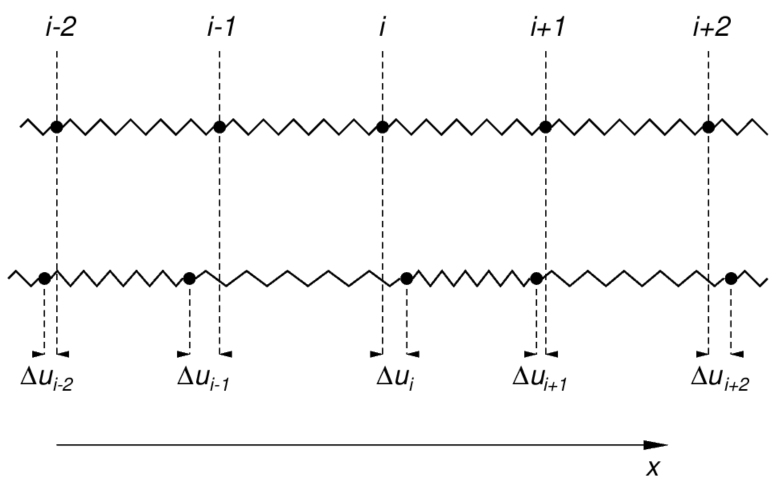

Figure 1.2: A long line of "beads" of mass m each, connected by identical springs. The chain is parallel to an x coordinate axis, and the beads are numbered with consecutive numbers i. The plot shows a fragment of the chain, far away from each end. In the upper "chain" all springs are unstreched, and all beads rest in their equilibrium positions, symbolized by the long vertical dashed lines. The lower chain is an instantaneous situation when all beads are moving, and Δu_i is the displacement of the i-th bead from its equilibrium position.

Consider the forces acting on the i^{th} bead. A displacement $\Delta u_i > 0$ compresses the spring on its right side, whereas a displacement $\Delta u_i < 0$ stretches this spring. Conversely, a displacement of the $(i+1)$-th bead stretches the same spring if $\Delta u_{i+1} > 0$, or compresses it, if $\Delta u_{i+1} < 0$. Therefore, the net force F_{R} exerted on the i-th bead by the right-side spring can be written as:

$$F_{\text{R}} = -k\Delta u_i + k\Delta u_{i+1}$$

From a similar analysis performed for the force exerted on the i^{th} bead by the left-side spring one gets:

$$F_{\text{L}} = -k\Delta u_i + k\Delta u_{i-1}$$

So, the net force acting on the i^{th} bead is:

$$F_{\text{net}}(i) = k(\Delta u_{i-1} - 2\Delta u_i + \Delta u_{i+1}) \tag{1.20}$$

Suppose that the distance between successive beads is Δx. Now we can do a small trick with the right side of the above equation, and multiply it by something that is sometimes called a"well-chosen unity", namely, by $(\Delta x)^2/(\Delta x0^2 = 1$, to get:

$$F_{\text{net}}(i) = k \cdot (\Delta x)^2 \left[\frac{\Delta u_{i-1} - 2\Delta u_i + \Delta u_{i+1}}{(\Delta x)^2}\right] \tag{1.21}$$

We can also invoke the Newton's Second Law of Dynamics: $F = m \cdot a$, where a is the acceleration. Hence, the acceleration a_i of the i^{th} bead in the chain can be expressed as:

$$a_i = \frac{k \cdot (\Delta x)^2}{m} \left[\frac{\Delta u_{i-1} - 2\Delta u_i + \Delta u_{i+1}}{(\Delta x)^2}\right] \tag{1.22}$$

Note the similarity of the expression in square brackets [...] to the right side of the Eq. 1.6, which in the $\Delta x \to 0$ limit becomes the second derivative of $f(x)$. Now, we can think of a similar operation on Eq. 1.22. Physically, it will mean that the distance Δx between the beads gets shorter and shorter, down to the atomic scale – in the limit, there are no beads, but atoms, connected by atom-atom interaction forces (in fact, for a relatively small change of the atom-atom distance, such forces are pretty well described by the Hooke's Law). So, from a "macro perspective", the displacement u can be treated as a smooth function of x. On the other hand, the acceleration of a bead is the second derivative of its displacement with respect to time. It means that the "smooth" displacement function u should be treated as a function of both x and t: $u = u(x, t)$. Then, the Eq. 1.22 can be rewritten in terms of a second-order derivatives:

$$\frac{\partial^2 u}{\partial t^2} = \frac{k \cdot (\Delta x)^2}{m} \cdot \frac{\partial^2 u}{\partial x^2} \tag{1.23}$$

In order to make the equation more compatible with the "macro word", let's assume that the chain considered consists of N "beads", and its total mass is M, and its total length is L. Then, $m = M/N$, and $\Delta x = L/N$. The coefficient in front of the right-hand derivative can be therefore transformed to:

$$\frac{k \cdot (\Delta x)^2}{m} = \frac{k \cdot \frac{L^2}{N^2}}{\frac{M}{N}} = \frac{\frac{k}{N} L^2}{M}$$

There is still the "micro" parameter k we want to replace by a "macro" one. Which is not a big problem! The Hooke's Law states that for a single spring between two "beads" a force F leads to a displacement Δu, such that: $F = k \cdot \Delta u$, so that $k = F/\Delta u$. If the same force is applied to a chain consisting on N such spring, each will elongate by Δu, so that the total elongation of the chain will be $N \cdot \Delta u$, and the overall "spring constant" K for the chain will be:

$$K = \frac{F}{N \cdot \Delta u} = \frac{k}{N}$$

Putting everything together into the Eq. 1.23, we obtain the following equation of motion:

$$\frac{\partial^2 u}{\partial t^2} = \frac{K \cdot L^2}{M} \cdot \frac{\partial^2 u}{\partial x^2} \tag{1.24}$$

You may ask, why don't we use the symbols d^2u/dt^2 and d^2u/dx^2 for the second-order derivatives, but $\partial^2 u/\partial t^2$ and $\partial^2 u/\partial x^2$? Well, the answer is that the displacement u is a function of **two** variables, $u = u(x, t)$. And for functions of two or more variables, a derivative with respect to any of these variables is called a ***partial derivative***, and the d symbols in such derivatives are replaced by ∂. In order to calculate a partial derivative with respect to one of the variables, one treats all other variables as constants.

The Equation 1.24 is usually referred to as the ***Wave Equation***. It was derived originally by the French mathematician, physicist and astronomer

Jean-Babtiste d'Alembert more than 250 years ago. The d'Alembert's Equation is not only valid for beads chains – it applies to a number of other physical systems, only the coefficients in front of the $\partial^2 u/\partial x^2$ are different in each case. In particular, the d'Alambert Equation describes sound waves (more about this equation can be found in this Wikipedia article).

In order to find the solution of the d'Alembert Equation for the sound waves, we will first consider a "generic" equation of the form:

$$\frac{\partial^2 u}{\partial t^2} = C^2 \cdot \frac{\partial^2 u}{\partial x^2}, \tag{1.25}$$

treating C^2 as a constant, but not yet specifying what physical sense it has. Why C^2, and not just C? It will become clear in a moment.

iddle

1.3.1 Solving the General Wave Equation

As noted, there are no simple rules of solving differential equations (DEs) – no "universal algorithms" as in the case of many different types of algebraic equations[1] The solutions of differential equations are not numbers, but **functions**. The process of solving a DE is more like solving a riddle. Intuition, experience, plus a bit of common sense are helpful.

Lets try a "test solution" of the form:

$$u(x,t) = A\sin(\xi \cdot x + \zeta \cdot t) \tag{1.26}$$

where A, ξ and ζ are constants. Why are they needed? Well, note that the displacement function, $u(x,t)$, is not a dimensionless number – it must be expressed in *units of length*, and in this course we use *meters* in this role. And the sine function is by definition unit-less, so that it has to be multiplied by a constant expressed in meters. Since the sine function can only take values from the $[-1,+1]$ range, A determines the maximum and the minimum value the solution function may take – we call it the *amplitude*, as in the case of a simple oscillator.

Second, not only sine function itself, but also its *argument* must be a dimensionless number. And both x and t are expressed in units, respectively, of length and time. So, they have to be multiplied by constants that

[1] For instance, such algorithm for a quadratic equation, $\alpha x^+ \beta x + \gamma = 0$, is, as you may remember from high school math classes: $x_{1,2} = (-\alpha \pm \sqrt{\beta^2 - 4\alpha\gamma})/2\alpha$. Another example are the algorithms for solving systems of N linear equations with N unknowns.

"neutralize" their dimensions. So, for ξ let's choose a constant of the form: $\xi = 2\pi/\lambda$, with λ in meters. The sine is a periodic function with a repetition period of 2π – meaning that for any α value and any integer n

$$\sin(\alpha) = \sin(\alpha + 2\pi) = \sin(\alpha + 4\pi) = \ldots = \sin(\alpha + 2n\pi).$$

So, if x is multiplied by $2\pi/\lambda$, then λ becomes the repetition period along the x direction:

$$\sin\left[\frac{2\pi}{\lambda}x\right] = \sin\left[\frac{2\pi}{\lambda}(x+\lambda)\right] = \sin\left[\frac{2\pi}{\lambda}(x+2\lambda)\right] = \ldots = \sin\left[\frac{2\pi}{\lambda}(x+n\lambda)\right]$$

In a similar fashion, we can choose the ζ in the "test solution" in the Eq. 1.26 as $\zeta = 2\pi/T$. Then T becomes the period of the test solution function *in time:*

$$\sin\left[\frac{2\pi}{T}t\right] = \sin\left[\frac{2\pi}{T}(t+T)\right] simplyan = \sin\left[\frac{2\pi}{T}(x+2T)\right] = \ldots = \sin\left[\frac{2\pi}{T}(x+nT)\right].$$

OK, so now we have to check whether the proposed "test solution" is indeed a solution of the Equation 1.25. In order to do that, one extra thing is needed – namely, the rules of calculating the second-order partial derivatives of the sine function of two variables (where α, β and γ are arbitrary constants):

$$f(x,t) = \alpha \cdot \sin(\beta x - \gamma)t$$

$$\frac{\partial f}{\partial x} = \alpha\beta\cos((\beta x - \gamma t); \qquad \frac{\partial^2 f}{\partial x^2} = -\alpha\beta^2\sin((\beta x - \gamma t)$$
$$\frac{\partial f}{\partial t} = \alpha(-\gamma)\cos((\beta x - \gamma t); \qquad \frac{\partial^2 f}{\partial t^2} = -\alpha\gamma^2\sin((\beta x - \gamma t)$$

Accordingly, for our "test sunction":

$$u(x,t) = A\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right), \tag{1.27}$$

after inserting it to the Eq. 1.25, the left side of the equation becomes:

$$L = -A\frac{4\pi^2}{T^2}\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right),$$

and the right side yields:

$$R = -AC^2 \frac{4\pi^2}{\lambda^2} \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right),$$

So, $L = R$ when

$$\frac{1}{T^2} = \frac{C^2}{\lambda^2}, \quad \text{or} \quad C = \frac{\lambda}{T}$$

Note that the C coefficient is fully defined by the chain properties: $C^2 = K \cdot L^2/M$.

Now, we can rewrite the Eq. 1.27 in the form:

$$u(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - C \cdot t)\right], \tag{1.28}$$

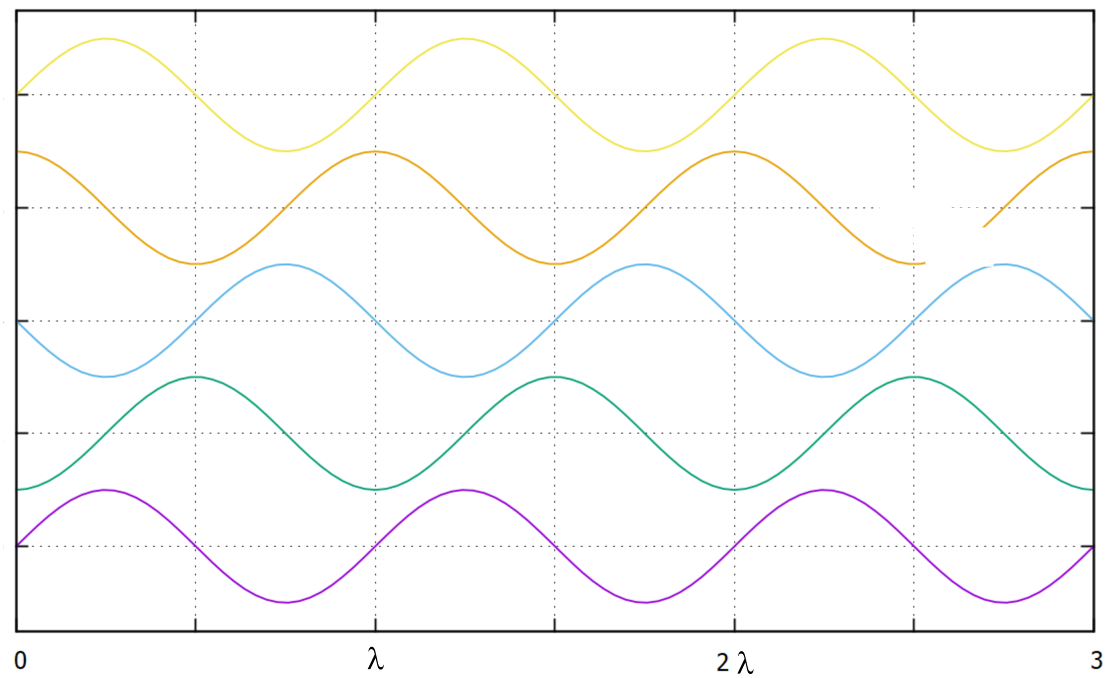

Figure 1.3: Plots of $u(x,t)$ given by the Eq. 1.28, as a function of x, for several different values of time t. The curves, from the bottom to the top, are plots of the $u(x,t)$ function for the values of the $C \cdot t$ term equal, respectively, to $C \cdot t = 0$, $\frac{1}{4}\lambda$, $\frac{1}{2}\lambda$, $\frac{3}{4}\lambda$, and λ. The plots shows that the $u(x,t)$ function describes a *propagating wave*. The "crests" of the wave gradually shift to the right. One can see that the crests of the uppermost wave (corresponding to $C \cdot t = \lambda$) are shifted to the right by λ with respect to the lowermost wave (for which $C \cdot t = 0$). In other words, when t increases from 0 to λ/C, the crests shift by λ to the right. The definition of speed is the distance traveled over the time it takes: so, the crests travel a distance of λ over a time period of λ/C – therefore, the speed with which the crests travel is $v = \lambda/(\lambda/C) = C$.

As explained by the Fig. 1.3 and the discussion in its caption, the solution of the d'Alembert Equation describes a propagating wave. The λ parameter has the physical meaning of the**wavelength**, i.e., the distance between two adjacent "crests" (or, equivalently, between two adjacent "troughs"). And the C parameter has the physical meaning of the *speed of propagation* of the wave, i.e., the speed with which the "crests" and the "troughs" propagate along the x axis. It has to be kept in mind that C **is not the speed of the constituent particles of the medium in which the wave propagates!** The said particles only execute an *oscillatory motion*, about the equilibrium position, which does not change.

The C is being used as the symbol of the speed of propagation of some wave types – most notably, light waves (but lower-case c rather than capital C). For sound waves, the much more often used symbol for "the speed of sound" is v – and this symbol will be used from now on in this document.

Let's then summarize: the "equation of a sound wave", i.e., the formula describing the oscillations in space and time that make a sound wave, is the following:

$$\boxed{u(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - v \cdot t)\right] \quad (1.29)}$$

The time it takes the "crests" to travel the distance of a single wavelength is λ/v, and it is equal to the elementary oscillation period T of the particles comprising the medium. So:

$$\boxed{T = \frac{\lambda}{v} \quad (1.30)}$$

In sound waves we deal with (speach, music, etc.) the air particles makes several hundred or even several thousand oscillations per second, so that the T periods for such waves are very small numbers. It is much more convenient to use the *frequency* f, which is the number of oscillations made in a single second. The relation between f and T is therefore:

$$\boxed{f = \frac{1}{T} = \frac{v}{\lambda} \quad (1.31)}$$

By combining the Eq. 1.29 with the Eq. 1.30 or Eq. 1.31, the Eq. 1.29 may be written in an equivalent form, without using the v parameter, but in terms of both the periodicity in space (λ) and the periodicity in time (T):

$$\boxed{u(x,t) = A \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) \quad (1.32)}$$

Or using the frequency instead of the time period:

$$u(x,t) = A\sin\left(\frac{2\pi}{\lambda}x - 2\pi f \cdot t\right) \quad (1.33)$$

Finally, physicists are not particularly fond of the "ordinary frequency" f, they prefer to multiply it by 2π, and they call the product "angular frequency" and use the ω symbol for it: $\omega = 2\pi \cdot f$. In terms of ω, the Eq. 1.33 can be rewritten as:

$$u(x,t) = A\sin\left(\frac{2\pi}{\lambda}x - \omega \cdot t\right) \quad (1.34)$$

This is not yet the end! One point still needs to be addressed. Why there is minus in the argument of the sine function in the Eqs. 1.29, 1.32, 1.33 and 1.34? If we put "+" instead, would such equations be no longer legitimate solutions of the d'Alembert Equation? The answer is "no" – the formula:

$$u(x,t) = A\sin\left(\frac{2\pi}{\lambda}x + \omega \cdot t\right) \quad (1.35)$$

is a solution of the d'Alembert Equation **as good as** the formula in Eq. 1.34.

Really? No difference? No, there is a difference. The Eq. 1.34 describes a wave propagating **from the left to the right** – as shown, e.g., in the Fig. 1.3. And the Eq. 1.35 is a formula for a wave propagating in the opposite direction, from the right to the left. If someone does not believe me – it is not difficult to check!

1.4 More About Waves

Let's recall the equation for time-dependent displacement in a simple harmonic[2] oscillator: but now we "rename" the osc illation direction to y (x will

[2]Not all simple oscillators are harmonic, i.e., such that their displacement vs. time is described by the $\sin(\omega t)$ function. There is a whole variety of oscillators that are simple, but not harmonic: we call them ***anharmonic***. For instance, a simple pendulum is a SHO

be reserved for other purposes):

$$\Delta y(t) = A\sin(\omega \cdot t)$$

But we will now return to an earlier version in terms of the oscillation ***period*** T:

$$\Delta y(t) = A\sin\left(\frac{2\pi}{T} \cdot t\right) \tag{1.36}$$

About this function, one can tell that it describes a ***displacement periodic in time***. Physicists also like the word ***perturbation***, one of its meanings is a deviation or a displacement from a state of equilibrium or a normal course of action. So, a deviation of an oscillator relative to the equilibrium point can also be called "a perturbation". Some people even prefer it, because it sounds "more professionally". Well, perhaps, but we will keep using the "deviation". Anyway, is you at some moment hear a person talking about a "perturbation" in oscillatory or wave motion, you will know what the person has in mind. Now, think of a chain of beads strung on a string, as in Fig. 1.4:

Figure 1.4: Beads on a stretched string.

Then, let someone grabs the left end of the string and starts wiggling it up and down – such an action will create a wave (physicists often say: "will excite a wave") propagating towards the other end:

oscillator as long as the displacement amplitude is small (say, 2-3°, but if it swings as far as 90° from the vertical axis, it becomes strongly anharmonic.

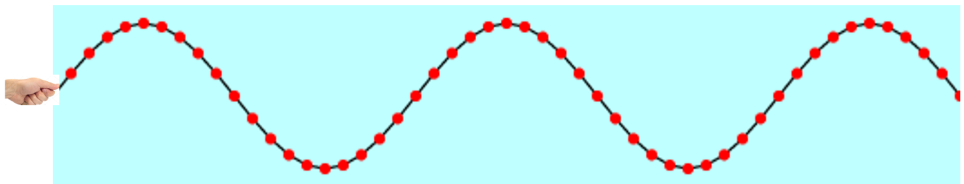

Figure 1.5: Exciting a wave in the bead chain.

It's clear that such an action created a state ***periodic in space*** – along the horizontal direction, call it x, or the ***propagation directio***.

The wave period along this direction – the distance between two consecutive maxima (usually, called "crests"), or two consecutive minima (often called "troughs") is referred to as the ***wavelength***, and conventionally denoted as λ (the lower-case Greek character "lambda"; upper case is Λ, the Roman equivalent is L).

By analogy to the function Eq. 1.36 that describes a periodicity in time, we can write a function describing the periodicity in space – just by replacing the time period T by the period in space λ, and the time variable t in the function's argument by the spatial variable x:

$$\Delta y(x) = A \sin\left(\frac{2\pi}{\lambda} \cdot x\right) \tag{1.37}$$

OK, but we need a function which is periodic ***both in time and space***, right? How we can get it? Well, it's simple: insert into the argument of the function in the Eq. 2.14 a second term corresponding to time periodicity (note that the displacement in the y direction becomes now a function of both x and time, $\Delta y = \Delta y(x, t)$):

$$\Delta y(x, t) = A \sin\left(\frac{2\pi}{\lambda} \cdot x - \frac{2\pi}{T} \cdot t\right). \tag{1.38}$$

This expression is often called a *wave function*, or a *waveform* – the latter term is more often used in electrical engineering, while the former one is more often used by physicists.

Conventionally, in the wave function's argument the "spacial" term with x goes first, and the term with the time variable t goes as the second. But why did I put the minus sign, and not a plus sign in between?

Well, I can put the plus sign, if you wish! Here you go! It's also a perfectly correct wave equation:

$$\Delta y(x,t) = A \sin\left(\frac{2\pi}{\lambda} \cdot x + \frac{2\pi}{T} \cdot t\right). \tag{1.39}$$

Are both correct, indeed? So the sign makes no difference?

Well, it does: the function with the minus sign propagates from the left to the right, and the wave in the Eq. 1.16 with the plus sign travels from the right to the left. Usually, we think about moving from the left to the right as of a motion in the "positive" direction, and motion from the right to the left is a motion in the "negative direction". And we always prefer positive things over negative things, so that if we want just to write down a "generic wave equation", we usually write it with a minus sign.

Hurray!!! Chapeau bas! – as the French say if there is a moment worth celebration (it means: Hats off!). We have derived the simplest wave equation! But even though it is simple, it is very useful, and we will use it several times in this course.

But this is not the end – in fact, we may do several more operations on the equation. However, at this moment it becomes very instructive to "set things in motion", i.e., to start using **animated** versions of Figure 2.2. But, regretfully, your instructor (Dr. Tom, I mean) has not yet learned how to install animated pictures in PDF files. I will figure out, hopefully, but now I have no idea. Well, but it's not a big problem! I know how to put animations into Power Point files! So, please click on this link which will open a Power Point file in which I installed the animations needed, plus a text describing things, and, please, continue reading!

One more important thing before you open the Power Point – a definition. Namely, the argument of the oscillating part of the wave function is called the ***phase*** of the wave. In our notation, what oscillates is the sine function:

$$\sin\left(\frac{2\pi}{\lambda} \cdot x - \frac{2\pi}{T} \cdot t\right),$$

so that the phase in the present case is:

$$\frac{2\pi}{\lambda} \cdot x - \frac{2\pi}{T} \cdot t$$

Often an abbreviation $\Phi(x,t)$ is used, to save space and time needed for typing the whole phase, which may be even more elaborate than in the present case (e.g., in wave functions of two- or three-dimensional waves which we will be talking about in the next chapter).

In addition to the time- and space-dependent terms, in the phase there may be a constant term, which we can call, for instance, Φ_0. Then the whole expression for the phase becomes:

$$\frac{2\pi}{\lambda} \cdot x - \frac{2\pi}{T} \cdot t + \Phi_0 = \Phi(x,t) + \Phi_0$$

The constant term Φ_0 is sometimes called the *phase shift*. Which is not 100% logical if we talk about a **single** wave, because it only makes sense to talk about a shift if it is clearly defined, *relative to what* the shifted object is shifted. But the term makes much sense is we are talking about **two waves**, one of which has a constant term Φ_1 in its phase, and the other has a constant term Φ_2. Then, there obviously is a phase shift *between the two waves*, and its value then is $\Delta\Phi = \Phi_1 - \Phi_2$. If $\Phi_1 > \Phi_2$, we say that Wave One is *leading*, and if $\Phi_1 < \Phi_2$, we say that it is *lagging*.

1.5 The Interference of Waves.

What happens if two waves produced by two different sources meet one another? Well, the answer is simple, their wavefunctions (or waveforms, if you prefer that term) **add algebraically**. Physicists coined the term *interference* for such algebraic addition. Pretty straightforward!

There are certain special situations that need to be discussed in closer details. In particular, the role of the phase shift in iterference phenomena. We have mentioned the phase shift when we taled about simple oscillators (see the Eqs. 1.18 abd 1.19), but we have not yet touched this subject in the case of wave motion. In fact, for waves things look very much the same way as in simple oscillatory motion: if the wave function given by the equation Eq. 1.29, namely:

$$u(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - v \cdot t)\right]$$

is a good solution of the d'Alembert Equation, the fundamental equation governing the wave motion – then a function:

$$\boxed{u(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - v \cdot t + \Phi)\right] \quad (1.40)}$$

where Φ is any arbitrary number – positive or negative is a good solution as well. Similarly, one can add an arbitrary constant number Φ to the arguments of the sine function in all solutions equivalent to that in the Eq. 1.29:

$$u(x,t) = A \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t + \Phi\right),$$

$$u(x,t) = A \sin\left(\frac{2\pi}{\lambda}x - 2\pi f \cdot t + \Phi\right),$$

or

$$u(x,t) = A \sin\left(\frac{2\pi}{\lambda}x - \omega \cdot t + \Phi\right).$$

The Φ number is usually referred to as the *phase angle*. If we consider a single wave, then the phase angle is usually irrelevant and one can just drop it (which is the same as assuming that $\Phi = 0$. However, if there are two or more waves – e.g., in interference phenomena – the the phase angles of participating waves are of crucial importance! In fact, not just the angles themselves, but the difference between them, commonly referred to as the "phase shift". In the interference of waves phenomena the phase shifts play a principal role!

There are certain special cases of interference, that are particularly interesting – not only in their own right, but, in addition, due to the role they play in practical devices. Here we will present two such cases. One is the occurrence of the so-called ***beats***. They may ruin an orchestra performance, so that musicians hate them! The other is the creation of the so-called ***standing waves***. They are of crucial importance, not only in acoustic phenomena, but also in physics of light, of electromagnetic waves, and even in physics of seismic waves created by earthquakes – and this is not the end of the list.

1.5.1 Constructive and Destructive Interference

Constructive interference between two waves occur when there is no phase shift between two waves – or, the phase difference is $\Phi_1 - \Phi_2 = (2\pi) \times n$, where n is an integer number (if the Φ angles are expressed in *radians*; alternatively

– for those who prefer to use degrees, not radians – if the phase difference is an integer multiple of 360°: $\Phi_1 - \Phi_2 = n \times 360°$).Such a situation is presented in the Fig. 1.6; an often used term is: *the two waves are* ***in-phase***. When the constructive interference condition is satisfied, the crests of the two waves occur at the same positions at the x axis, as well as the troughs – as the result of the algebraic addition of the two waves, the amplitude of the resultant wave is the sum of the amplitudes of the two waves.

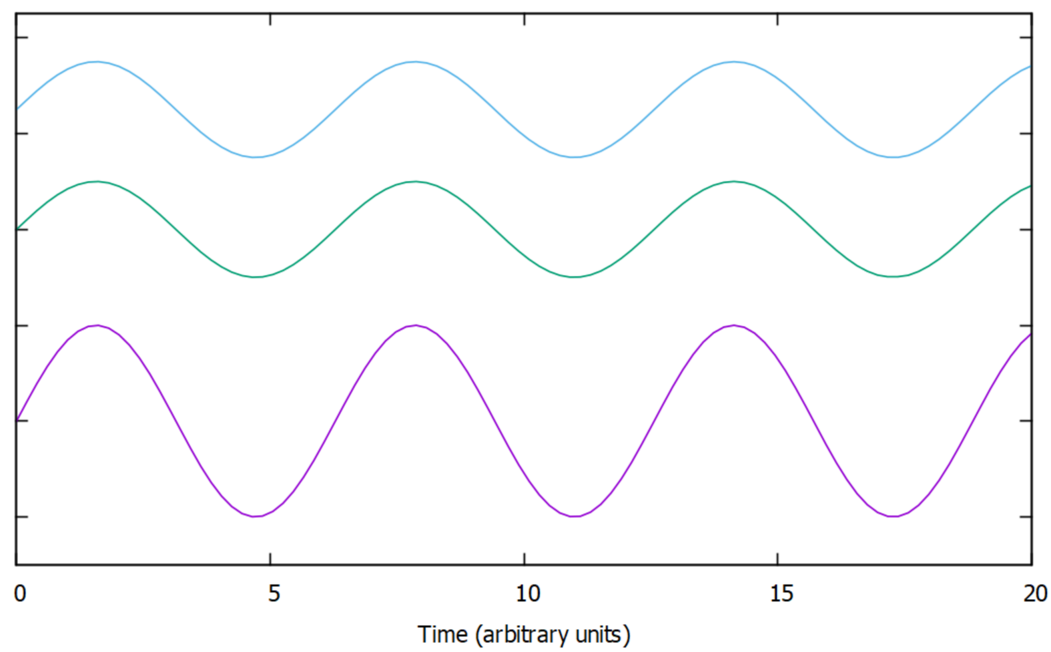

Figure 1.6: The upper two sinusoids represent two waves that are *in phase*. The lower-most curve is the wave created by the interference of those two waves.

If the phase difference between two waves is π, 3π, $5\pi, \ldots$ – or 180°, 540°, 720°, ... – i.e., the *odd multiple* of π, or of 180° – the crests of one wave occur at the same position as the troughs of the other wave, and *vice versa*. A commonly used term for such a situation is: *the waves are* ***out of phase***. As the result, if the two waves are of the same amplitude, they completely cancel out. Here the interference is ***destructive***. Such a situation is illustrated in the Fig. 1.7.

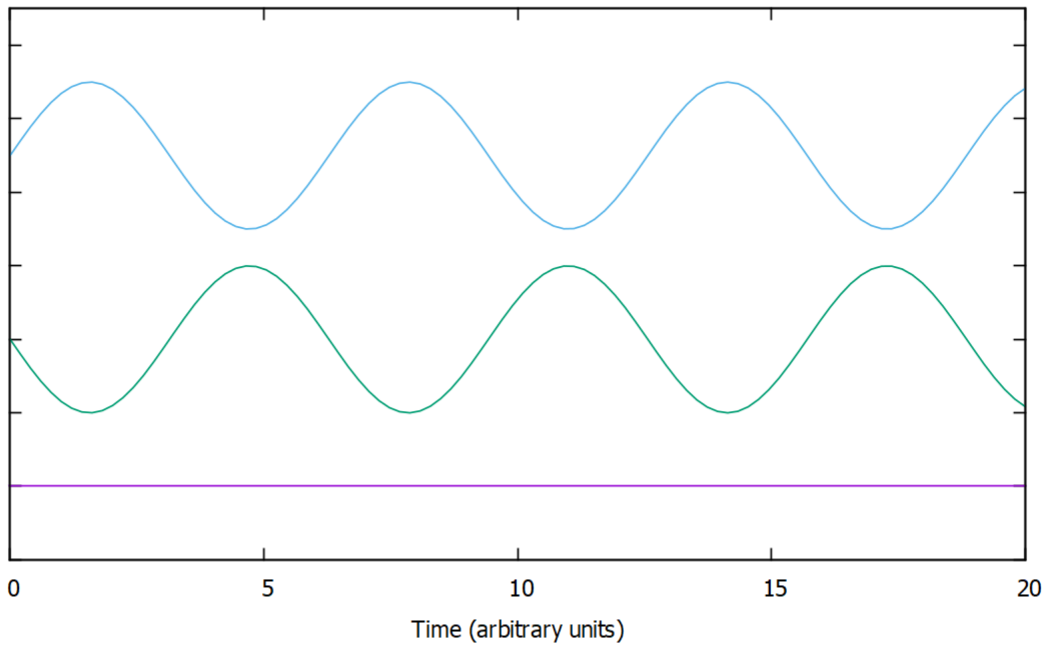

Figure 1.7: The upper two sinusoids symbolize two waves that are *out of phase*. They cancel one another, which is symbolized by the flat line at the bottom of the plot.

1.5.2 Beats

An interesgting effect is produced by the interference of two waves of slightly different frequencies f_1 and f_2. The interference between produces an alternating pattern of constructive-destructive-constructive-destrucive-... regions. If the two waves are sound waves, then what a person hears is a modulated sound – an effect that can be readily demonstrated in classroom, and the students usually enjoy such a show.

The mechanism producing "beats" is explained in the Fig. 1.8. The two upper sinusoids symbolize two waves with different frequencies – as can be seen in the plot, at the beginning of the time scale the two waves are "in-phase". However, in the time period between 0 and 60 on the horizontal scale, the upper "blue" wave makes 11 full cycles, and the other "green" wave makes only 10 full cycles. So, ratio of the frequencies of the two waves, f_1 and f_2, respectively, is $f_1/f_2 = 11/10 = 1.1$.

Therefore, at $t = 60$ the two waves are again "in-phase" – but half-way between $t = 0$ and $t = 60$ they are precisely "out-of-phase" (the frequencies of the two waves were chosen in such a manner that the strongest destruction interference and the strongest construction interference occur at "round" t values, respectively: at $t = 30$, 90, and 150, and at$t = 60$, 120, and 180).

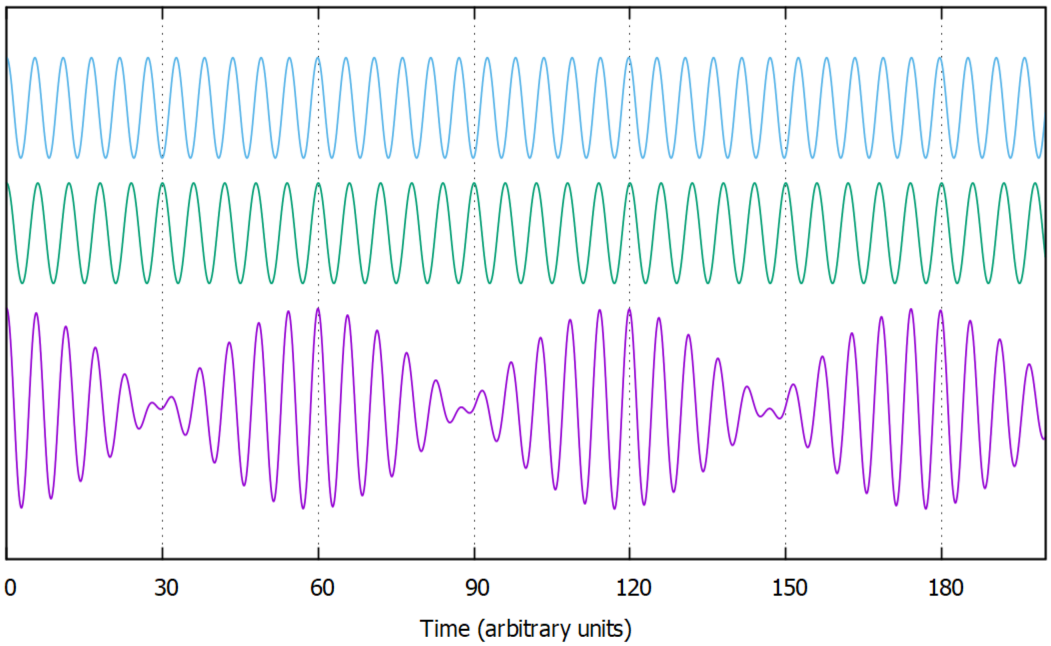

Figure 1.8: Explanation of the interference mechanism producing "beats" (for details, see the text).

In order to show the phase shift at $t = 30$ and $t = 60$ more clearly, in the Figs. 1.9 and 1.10 these regions are shown with the t scale "blown-up" (i.e., stretched) 10 times.

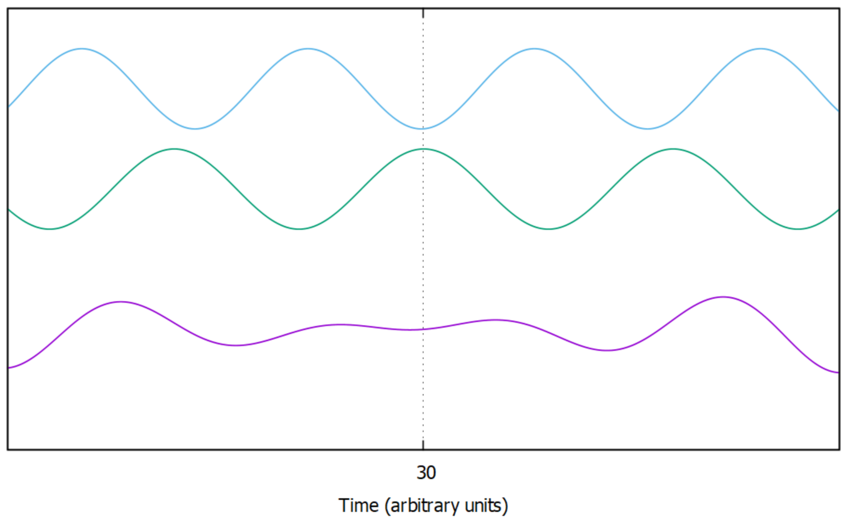

Figure 1.9: Destructive interference at $t = 30$, with the time scale stretched 10 times compared to that in the Fig. 1.8.

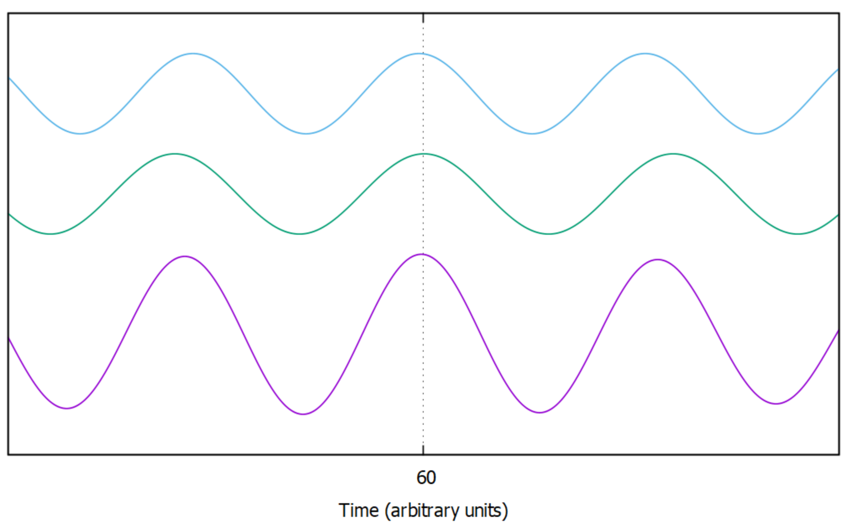

Figure 1.10: Constructive interference at $t = 60$, with the time scale stretched 10 times compared to that in the Fig. 1.8.

The graphic explanation of the mechanism producing beats is very instructive (at least in Dr. Tom's opinion!), but a mathematical explanation is certainly pretty instructive, too. So, let's consider two waves with frequencies f_1 and f_2, such that $|f_1 - f_2| \ll f_1$ (you know the symbol $<$, meaning *is smaller than) – the symbol ll means is **much** smaller than.*

So, we will consider the interference of two waves – it will be most convenient to use the wavefunctions in the form given by the Eq. 1.33:

$$u_1(x,t) = A\sin\left(\frac{2\pi}{\lambda_1}x - 2\pi f_1 \cdot t\right)$$

and

$$u_2(x,t) = A\sin\left(\frac{2\pi}{\lambda_2}x - 2\pi f_2 \cdot t\right)$$

Here we can get the wavelengths in terms of the speed of sound v and frequency, using one of the famous "Dr. Tom's triangles"– $\lambda_1 = v/f_1$, and $\lambda_2 = v/f_2$ – but we don't need to do that. In order to make the calculations easier, let's simply assume that we are located at a point for which the x coordinate is $x = 0$ (in the case of coordinate systems, one has a total freedom of deciding where the *origin* is – so, taking advantage of this liberty, let's decide that the origin is located precisely at the same point at which we are positioned, so that our x coordinate is zero. Then the formulas for the two waves reduce to:

$$u_1(x,t) = A\sin\left(-2\pi f_1 \cdot t\right) \qquad \text{and} \qquad u_2(x,t) = A\sin\left(-2\pi f_2 \cdot t\right)$$

In order to find the algebraic sum of the two, we can use the trigonometrical identity:

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

After applying this charming (isn't it so?) formula, we get (for brevity, I'll write simply u_1 instead of $u_1(x,t)$, and the same for the other guy):

$$u_1 + u_2 = 2A\sin\left(-2\pi\frac{f_1+f_2}{2}t\right)\cos\left(2\pi\frac{f_1-f_2}{2}t\right)$$

Which, putting $f_{\text{aver.}}$ for $\frac{1}{2}(f_1+f_2)$, we can rewrite as:

$$u_{\text{net}}(t) = \left[2A\cos\left(2\pi\frac{f_1-f_2}{2}t\right)\right] \times \sin\left(2\pi f_{\text{aver.}} \cdot t\right) \qquad (1.41)$$

The expression in between the square brackets [...] can be thought of as the slowly-oscillating "envelope" for the fast-oscillating function $\sin(2\pi f_{\text{aver.}} t)$.

So, the frequency of the "beats" we hear is equal to the difference between f_1 and f_2, the frequencies of the two sources responsible for creating the effect. We usually write $f_{\text{beats}} = |f_1 - f_2|$. If you hear beats, there is no way of figuring out which of the two frequencies is higher, and which is lower – only the *absolute value* of the difference is important. If one wants to know whether $f_1 > f_2$, or $f_1 < f_2$, extra measurements are needed (e.g., using a device capable of measuring each individual frequency with sufficient precision).

1.5.3 Standing Waves

A highly important effect commonly referred to as a "standing wave" is created by the interference of two waves of *exactly the same frequencies*, but *propagating in* ***opposite directions***. Let's recall what was said a moment ago: if the equation describing a wave is:

$$u(x,t) = A\sin\left(\frac{2\pi}{\lambda}x - 2\pi f\cdot t\right),$$

then the expression (note the change of the minus sign to a plus sign):

$$u'(x,t) = A\sin\left(\frac{2\pi}{\lambda}x + 2\pi f\cdot t\right)$$

is the equation of a wave of exactly the same frequency, wavelength and amplitude, propagating in the opposite direction.

Let's check what is the result of the interference of two such waves – let's add them algebraically:

$$u(x,t) + u'(x,t) = A\sin\left(\frac{2\pi}{\lambda}x - 2\pi f\cdot t\right) + A\sin\left(\frac{2\pi}{\lambda}x + 2\pi f\cdot t\right) \quad (1.42)$$

Here the trigonometric identity we have used a moment ago:

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

comes in handy and we obtain:

$$u(x,t) + u'(x,t) = 2A\sin\left(\frac{2\pi}{\lambda}x\right) \times \cos\left(2\pi f\cdot t\right) \quad (1.43)$$

The cosine in the above equation is simply an oscillating function, depending on time only, but not on x. And the part to the left of the multiplication sign $\times$, namely:

$$2A\sin\left(\frac{2\pi}{\lambda}x\right)$$

can be thought of as the **amplitude** of such oscillation, modulated along the x axis – or, a better name may be the *x-depending modulating function.* Note that this modulation function periodically takes the value of zero, for the $2\pi x/\lambda$ argument values equal:

$$\frac{2\pi}{\lambda}x = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ldots$$

which correspond to the x values of:

$$x = 0,\ \frac{1}{2}\lambda,\ \lambda,\ \frac{3}{2}\lambda,\ 2\lambda,\ \frac{5}{2}\lambda,\ldots$$

So, at certain values of x, *half a wavelength apart*, the oscillations are ***totally suppressed***. These x points are called the ***nodes*** of a standing wave. Why standing? Because the nodes are *fixed*, they do not move along the x axis!

In contrast, for the argument values:

$$\frac{2\pi}{\lambda}x = \frac{1}{2}\pi,\ \frac{3}{2}\pi,\ \frac{5}{2}\pi,\ \frac{7}{2}\pi\ldots$$

which correspond to the x values of:

$$x = \frac{1}{4}\lambda,\ \frac{3}{4}\lambda,\ \frac{5}{4}\lambda,\ \frac{7}{4}\lambda,\ \frac{9}{4}\lambda,\ldots$$

i.e, for x points $\lambda/2$ apart, located exactly *half-way between two adjacent nodes*, the modulation function attains the value of $+2A$ or $-2A$, so the displacement at such points is the strongest – they are called the ***antinodes***. Note that the positions of the antinodes, like the positions of the nodes, are fixed. Therefore, the result of the interference of the two waves resembles a wave, but a wave that is "not moving" – therefore, it's called a***standing wave***.

A very instructive piece of animated graphics explaining how a standing wave is created, can be found in this Wikipedia article. A good idea is to click on the animation to the right of the top of the text. Then it opens, much larger, on a separate page. It is good to notice that at x points where the nodes form, the two interfering waves – the "blue" one and the "green" one – at any instant of time are out-of-phase, so that they cancel each other. Contrary, at x locations where the antinodes form, at any time instant the two waves are in-phase, so that they always interfere constructively – they always "augment one another".

1.5.4 Standing Waves on Stretche Strings

If a wave is created in a uniform stretched string (physicist like to say that the wave is "excited"), then it propagates with the speed:

$$v = \sqrt{\frac{F}{\mu}} \tag{1.44}$$

where F is the tension force in the string, and μ is the mass of the string for a unit length – if the length of the string in question is L, and the mass of this particular piece of string of length L is m, then $\mu = m/L$. Please check that if F is in Newtons, and μ is in kilograms per meter, then from the right side of the the Eq. 1.44 you indeed get a result in expected units, i.e., in meters per second.

As I told you not only once and not only twice, I hate to show my students a piece of math and tell them: *This is the formula for .... – believe me!*. I always want to show how the formula in question is derived from "the first principles". But now I will not present the math in this chapter – instead, I will give you the links two short YouTube presentation (Video One, 4 min. 49 s., and Video Two, 15 mins.), in which the Eq. 1.44 is derived using two slightly different methods.

How, then, would it be possible to create a standing wave in a stretched string? The answer is simple: send one wave of frequency f_1 from one end, toward the other end. And a wave of the same frequency from the other end. Well, it seems simple, but it's impractical, because the two waves must be identical: the slightest difference in the frequencies of the two waves will cause the nodes and the antinodes to move, so that there will be no "standing" wave!

There is a simple solution to the problem – send only one wave wave from one end, and immobilize the oththe frequency er end – by fixing it to a wall, or to a heavy object. A wave propagating towards this end will be “back-reflected”, and will start to propagate in the opposite direction. The back-reflected wave will have the same frequency as the original one – the two will be identical, only propagating in opposite directions. So, we will get what is needed for creating a standing wave!

Well, it's not exactly so simple. It would not work for all frequencies. Note that if both ends of the string are fixed, they cannot move – the only standing waves that may occur in such a string are those which have a *node at each end*. The simplest case of such a a standing wave is a one with nodes at each end, and an anti-node in the middle. As we said, nodes are $\lambda/2$ apart. So, in the case considered the two nodes are L apart, so that the wavelength λ of the standing wave must be $2L$. Good! We know the propagation speed v, we know the λ value – and we remember that the relation between λ, v, and the frequency f is $f = v/\lambda$ (one of the “Dr. Tom's triangles”). So, we can conclude that the frequency of the standing wave with nodes at the two string ends, and an anti-node in between, must be:

$$f = \frac{v}{\lambda} = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \tag{1.45}$$

However, this is not the only possible case. The limiting factor is *a node at each end*, but it does not forbid more nodes! The other possible situation is a node in the middle of the rope, and two anti-nodes, $L/4$ from each end. Then, the distance between the nodes is $L/2$, so that now the wavelength, call it λ_2, is $\lambda_2 = 2 \times L/2 = L$. The frequency of such a wave – call it f_2 – must be then:

$$f_2 = \frac{v}{\lambda_2} = \frac{1}{L}\sqrt{\frac{F}{\mu}} \quad \text{or} \quad f_2 = \frac{2}{2L}\sqrt{\frac{F}{\mu}} \tag{1.46}$$

But this is not yet the end! There may be a total of four nodes, two at the ends, and two located $L/3$ from each end, with a distance between them also being $L/3$. So, the wavelength will be $\lambda_3 = 2 \times L/3$, and the frequency will be:

$$f_3 = \frac{v}{\lambda_2} = \frac{3}{2L}\sqrt{\frac{F}{\mu}} \tag{1.47}$$

One can continue with such an analysis, and one can conclude that:

- In a stretched string, standing waves may form with a number of anti-nodes equal $n = 1,\ 2,\ 3,\ 4,\ 5, \ldots;$
- The number of nodes is $n + 1$, i.e., it's larger by one than the number of anti-nodes;
- The wavelength corresponding to a standing wave with n nodes is $\lambda_n = 2L/n$;
- The frequency f_n of a wave with n nodes is:

$$f_n = \frac{n}{2L}\sqrt{\frac{F}{\mu}}$$

The lowest frequency of a standing wave that may occur in a stretched string is then:

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \tag{1.48}$$

We call it the *fundamental frequency*. Note that the other frequencies f_n are simply the multiples of f_1, namely, $f_n = n \times f_1$. Such frequencies are called *harmonics*, f_2 is the *second harmonic frequency* or simply the *second harmonic*, f_3 is the *third harmonic*, and so on. The possible standing waves in the string are called *harmonic modes* or simply *modes*. First comes the *fundamental mode*, then goes the *second harmonic mode*, then the *third harmonic mode*, and so on.

EXPERIMENT

$L =$ .... m.

m	f_1	f_2	f_3	f_4

$L =$ .... m.

m	f_1	f_2	f_3	f_4

$L =$ .... m.

m	f_1	f_2	f_3	f_4

THEORY

$L =$ .... m.

m	f_1	f_2	f_3	f_4

$L =$ .... m.

m	f_1	f_2	f_3	f_4

$L =$ .... m.

m	f_1	f_2	f_3	f_4

CONCLUSIONS