Simple-minded Wave Packet tutorial

We will be summing ("superposing") simple waves of the form: \( f(x) = \sin(kx) \), with \( k \) values from a narrow range.

Let the \( k \)-range width be always 10% of the average \( k \) value. For simplicity, let the average value always be \( k_{\text{average}} = 1 \).

Let’s begin with just two waves. The two \( k \) values should be then: \( k_1 = 0.95 \) and \( k_2 = 1.05 \).

As you can see in the plot below, the resultant wave exhibits characteristic “groups” or “packets”.
Superposition of two waves

\[ \sin(0.95x) + 7 \]
\[ \sin(1.05x) + 4 \]
\[ \sin(0.95x) + \sin(1.05x) \]
Now, let’s take three waves. The $k$ values should be now: $k_1 = 0.95$, $k_2 = 1.0$, and $k_3 = 1.05$.

As you can see in the plot below, now the “packets” shift apart a bit. A smaller packet forms between each pair of large packets.
Superposition of 3 waves

- \( \sin(0.95x) + 11 \)
- \( \sin(1.0x) + 8 \)
- \( \sin(1.05x) + 5 \)

\( \sin(0.95x) + \sin(x) + \sin(1.05x) \)
Now, let’s take five waves, with $k_1 = 0.95$, $k_2 = 0.975$, $k_3 = 1.0$, $k_4 = 1.025$, and $k_5 = 1.05$.

Now the “main packets” shift apart even more, and more “baby packets” form in between.
Superposition of 5 Waves

\[
sin(.95x) + 20
\]

\[
sin(.975x) + 17
\]

\[
sin(x) + 14
\]

\[
sin(1.025x) + 11
\]

\[
sin(1.05x) + 8
\]

\[
sin(.95x) + sin(.975x) + sin(x) + sin(1.025x) + sin(1.05x)
\]
Next, take as many as **eleven** waves, with $k_1 = 0.95$, $k_2 = 0.96$, ..., $k_{10} = 1.04$, and $k_{11} = 1.05$. It’s difficult to show everything in a single plot – so let’s make two, one displaying the 11 waves, and the other the sum only.

The distance between the “main packets” is now even larger, and there is a whole bunch of “baby packets” in between.
11 waves $\sin(kx)$ with $k=0.95, 0.96, \ldots, 1.04, 1.05$
We may continue and keep adding more and more waves, and the distance between the “main packets” will further increase.

Note that the central packet – the one that forms around $x = 0$ – does not move. So, if the number of waves we take goes to infinity, only the central packet will remain. The next one to its right side will shift to plus infinity, and the next one to its left side will shift to minus infinity. But infinity means nowhere! So, by superposing an infinite number of constituent waves, we obtain a single wave packet!

However… there are still those annoying “baby packets”… Is there a way of eliminating them?
The answer is yes!
Note that until now the constituent waves we were taking were all of the same amplitude.

Let’s now take 11 waves with a Gaussian distribution of amplitudes. They are shown in the next plot. The wave with the middle $k$ value has the largest amplitude, and for larger and smaller $k$ values the amplitude gradually decreases. And as you can see in the plot showing the sum of those 11 waves, the “baby packets” are no longer there!
11 waves with Gaussian distribution of amplitudes
Superposition of 11 waves with Gaussian distribution of amplitudes
**Conclusion:** by adding up an infinite number of de Broglie waves from a narrow range of wavenumbers $k$, and with a Gaussian distribution of amplitudes, we can obtain a single wave packet representing a localized particle. This solves the “localization problem”!

The sum of an infinite number of waves can be replaced by an integral. In fact, we usually write functions representing wave packets in an integral form.

The wave packet formalism solves the “localization problem”. It also solves the “particle velocity problem”. In contrast to a single de Broglie wave, a packet of de Broglie waves propagates with a velocity that is consistent with the particle velocity (as for now, please accept this without a proof).