so that:
\[
\int_{-\infty}^{+\infty} |\psi(x)|^2 \, dx = 1
\]

We say that the $\psi(x)$ function must be *normalized* — and the above equation expresses this "normalization requirement" mathematically.

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**APPLICATIONS OF TISE:**

- We will now discuss the application of Schrödinger Equation to several simple quantum-mechanical situations (but "simple" does not mean that they are not important!!).

**FREE PARTICLE:**

The simplest of all conceivable quantum mechanical systems is certainly a free particle moving in the $x$ direction, with no forces whatsoever acting on it. $F=0$ means that the potential energy is constant: $U(x) = \text{const.}$
As you remember, in the case of potential energy, we can arbitrarily choose where \( U = 0 \).

**Question:** when we talk about the potential energy due to Earth gravity, we often choose one special point to be that with \( U = 0 \). Where is this point located?

So, for the "free-particle" we can set \( U = 0 \) and not worry any more about it!

Then, the TISE becomes:

\[
- \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi
\]

or

\[
\frac{d^2 \psi}{dx^2} = -k^2 \psi \quad \text{with} \quad k^2 = \frac{2mE}{\hbar^2}
\]

This is a well-known equation, with a general solution:

\[
\psi = A \sin(kx) + B \cos(kx)
\]

where \( A \) and \( B \) are arbitrary constants.

This is an equation of a wave with

\[
\lambda = \frac{k}{2\pi}
\]
The equation is satisfied for any value of \( k \) meaning that the energy is:

\[
k^2 = \frac{2mE}{\hbar^2} \quad \Rightarrow \quad E = \frac{\hbar^2 k^2}{2m}
\]

So, the free-particle equation describes essentially the motion of any a particle with any energy \( E \).

Now, note that for non-relativistic particles:

\[
E = K = \frac{mv^2}{2} = \frac{m^2 v^2}{2m} = \frac{p^2}{2m} \quad \Rightarrow \quad p = \hbar k
\]

and

\[
E = \frac{\hbar^2 k^2}{2m}
\]

But \( \hbar = \frac{\hbar}{2\pi} \) and \( k = \frac{2\pi}{\lambda} \)

So: \( p = \hbar k = \frac{\hbar}{2\pi}, \frac{2\pi}{\lambda} = \frac{\hbar}{\lambda}, \) or \( \lambda = \frac{\hbar}{\rho} \)

The latter is simply the de Broglie wavelength.

So, the solution produced simply a de Broglie wave, which is not surprising!

But there is a small problem... (not mentioned in the book!)
We said that $\psi(x)$ **must** be normalized!

And

$$\int_{-\infty}^{+\infty} |A\sin(kx) + B\cos(kx)|^2 \, dx$$

$$= \int_{-\infty}^{+\infty} [A^2\sin^2(kx) + B^2\cos^2(kx) + ABSin(2kx)] \, dx = \infty ! ! !$$

So, did our very first encounter with the TISE produce an **illegal** solution?

Fortunately, $\psi$ can be **rescued**!

Since $\psi(x) = A\sin(kx) + B\cos(kx)$ for **any** $k$ is a "good" solution, any **sum** of those solutions with different $k$ values is also a **good** solution. And by adding many such solutions, we can obtain a wave packet!

$\psi$:

\[ \text{infinite wave} \quad \int_{-\infty}^{+\infty} |\psi|^2 \, dx = \infty \]

$\psi$:

\[ \text{finite wave-packet (Gaussian)} \quad \int_{-\infty}^{+\infty} |\psi|^2 \, dx = \text{finite value.} \]
2. PARTICLE IN A BOX

Meaning: particle between two walls. The collisions with each wall are elastic, i.e., the particle is bounced back with no energy loss.

This is again a simple problem - but it has many important implications, and is used in many "serious" physical models.

We describe this situation in terms of potential energy:

\[ \begin{align*}
\text{infinite } U & \quad \text{infinite } U \\
U = 0 & \quad U = 0 \\
x = 0 & \quad x = L
\end{align*} \]

The particle cannot be in the regions \( x < 0 \) and \( x > L \), so \( \psi(x < 0) = 0 \) and \( \psi(x > L) = 0 \)
Only for $0 \leq x \leq L$ the wavefunction can be $\psi(x) \neq 0$.

The Schrödinger Eq. for the region between the walls is:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi$$

And, as was said before, the solution for $\psi$ is:

$$\psi = A \sin(kx) + B \cos(kx) \quad \text{with} \quad k = \frac{2mE}{\hbar^2}$$

However, now there are some restrictions imposed on $\psi$. First, $\psi(x)$ must be a continuous function. For $x < 0$, as we said, $\psi = 0$. So, the "inside" solution also must be zero at $x = 0$. And $\cos(kx)$ is one for $x = 0$. So, we conclude that $B$ must be zero.

This leaves us with:

$$\psi = A \sin(kx)$$

But, again, $\psi = 0$ for all $x > L$, so to assure the continuity of $\psi$, it must be:

$$\psi(L) = A \sin(kL) = 0$$
Meaning that:

$$kL = 0, \pi, 2\pi, 3\pi \ldots n\pi \ldots$$

$kL = 0$ would mean $\psi = 0$ everywhere, or no particle. So, it has to be rejected.
Other $kL$ values are O.K. So we get:

$$k = \frac{n\pi}{L} \quad \text{with} \quad n = 1, 2, 3, \ldots$$

and:

$$\psi = A \sin \left( \frac{n\pi}{L} x \right)$$

Still, we have to determine the value of $A$.
Here, the normalization condition becomes helpful:

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1 \quad \text{in the present case} \quad \int_0^L A^2 \sin^2 \left( \frac{n\pi}{L} x \right) dx = 1$$

Use a dummy variable $z = \frac{n\pi}{L} x$: $dx = \frac{L}{n\pi} dz$

$$\int_0^L A^2 \sin^2 \left( \frac{n\pi}{L} x \right) dx = \frac{A^2}{n\pi} \int_0^{n\pi} \sin^2(z) dz = \frac{A^2}{n\pi} \cdot \frac{n\pi}{2} = \frac{A^2 L}{2} = 1$$

So $A = \sqrt{\frac{2}{L}}$
Our wavefunction is then:

$$\psi = \begin{cases} 
\sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right) & \text{for } 0 \leq x \leq L \\
0 & \text{for } x < 0 \text{ and } x > L 
\end{cases}$$

But we still have to find the energy $E$.

This is easy:

$$k^2 = \frac{2mE}{\hbar^2} \quad \text{and} \quad k = \frac{n\pi}{L}$$

So:

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2} \quad \text{with } n = 1, 2, 3, \ldots$$

The lowest allowed energy is for $n = 1$:

$$E_1 = \frac{\hbar^2 \pi^2}{2mL^2}$$

The other possible energies can be simply written as:

$$E_n = n^2 E_1$$

So

$$E_2 = 4E_1, \quad E_3 = 9E_1, \quad E_4 = 16E_1, \ldots$$