CHAPTER 26

Electric Potential and Capacitance

26.1 POTENTIAL DUE TO POINT CHARGES OR CHARGE DISTRIBUTIONS

26.1 What is the electric potential energy of a charge \( q \) in an electric field due to a group of fixed charges?

Potential energy at a point \( P \) = minus the work done by the electric forces (or plus the work done against these forces) in taking an object (in this case the charge \( q \)) from a given reference point to point \( P \). PE has meaning only for conservative forces. The Coulomb force is conservative, having the same basic form as the gravitational force, except that it can be either repulsive or attractive. If the electric field is due to a single charge, \( q_1 \), and the reference point for potential energy is taken as \( r = \infty \), we have \( \mathbf{F} = q \mathbf{E} = \frac{kq_1 q}{r^2} \mathbf{r} \) (\( r \) points from \( q_1 \) to \( q \)), and in analogy to gravity, \( PE = U = \frac{kq_1 q}{r} \). (If \( q \) and \( q_1 \) have opposite signs, \( F \) is attractive and \( U \) is negative, as for gravity.) Since the work done by a vector sum of forces equals the sum of the works done by the individual forces, we have, for the field \( \mathbf{E} \) due to a series of point charges, \( q_1, q_2, \ldots, q_n \),

\[
\mathbf{F} = q \mathbf{E} = \sum_{i=1}^{n} \frac{kq_i q}{r_i^2} \mathbf{r}_i \quad \text{and} \quad PE = U = \sum_{i=1}^{n} \frac{kq_i q}{r_i}.
\]

26.2 (a) What is meant by the electric potential (often called absolute electric potential) at a point in space? (b) How is this related to the electric field?

(a) The electric potential is the electric potential energy (Prob. 26.1), with respect to infinity, of a unit positive charge. The unit of electric potential is the volt, where \( 1 \text{ V} = 1 \text{ J/C} \). (b) In the notation of Prob. 26.1, write \( V = U/q \) for the electric potential, then, by definition of electric potential energy, a small change \( \Delta s \) (in a specified direction \( s \)) in the position of a unit positive charge gives rise to the change

\[
\Delta V = -E_s \Delta s
\]

in the electric potential. In particular, for displacements parallel to the coordinate axes, (1) gives

\[
E_x = \frac{\Delta V}{\Delta x}, \quad E_y = -\frac{\Delta V}{\Delta y}, \quad E_z = -\frac{\Delta V}{\Delta z}
\]

Conversely, (1) follows from (2). Note that (1) implies the following relation among units: \( 1 \text{ V/m} = 1 \text{ N/C} \).

Define electric potential energy and electric potential using the precise language of calculus.

The electric force \( \mathbf{F} \) exerted on a test charge \( q' \) by some stationary distribution of charges is a conservative force. Therefore, the test charge possesses electric potential energy \( U \). With the reference point taken at infinity, the absolute potential energy of the test charge at the location \( B(x, y, z) \) is given by

\[
U(x, y, z) = -\int_{(x', y', z')}^{(x, y, z)} \mathbf{F} \cdot ds = -\int_{(x', y', z')}^{(x, y, z)} F_x dx + F_y dy + F_z dz \quad (1)
\]

with the integral taken along any path.

Conversely, if the electric potential energy is known as a function of position, then the component of the electric force along an arbitrary direction \( ds \) can be calculated as

\[
F_s = \frac{dU}{ds}
\]

In particular, the components of \( \mathbf{F} \) along the \( X, Y, \) and \( Z \) axes are given by

\[
F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}
\]

so that, in vector form,

\[
\mathbf{F} = \left( \frac{\partial U}{\partial x} \mathbf{i} + \frac{\partial U}{\partial y} \mathbf{j} + \frac{\partial U}{\partial z} \mathbf{k} \right) \quad (N)
\]

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If the only force acting on a point charge is the electric force $\mathbf{F}$, conservation of energy takes the form
\[ \Delta K + \Delta U = 0, \]
where, as usual, $K$ is the kinetic energy of the point charge.

The electric potential energy per unit test charge is called the electric potential, $\phi$, or the voltage, $V$. Since $\phi = U/q$ and $E = \mathbf{F}/q$, $\phi$ and $E$ enjoy the same relation as $U$ and $F$:
\[ \phi(x, y, z) = -\int_{(x, y, z)}^{(x', y', z')} E_x \, dx + E_y \, dy + E_z \, dz \quad (V) \]
\[ E = -\left( \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} \right) \quad (V/m) \]

26.4 What is meant by the potential difference between two points in space?

The potential difference from one point, $A$, to another point, $B$, is the work done against electric forces in carrying a unit positive test charge from $A$ to $B$. Thus the potential difference from $A$ to $B$ is $V_B - V_A = V$. Its units are those of potential, joules/coulomb = volts, and it is often called voltage or “voltage change.”

Because work is a scalar quantity, so too is potential difference. Like work, potential difference may be positive or negative.

The work $W$ done against electric forces in transporting a charge $q$ from one point, $A$, to a second point, $B$, is $W = q(V_B - V_A) = qV$.

26.5 Derive an expression for the potential due to a point charge.

If the source of the field is a point charge $q$ located at the origin (Fig. 26-1), then, the field at $P(x, y, z)$ is given by
\[ E' = \frac{q}{4\pi \varepsilon_0 r'^2} \]

We choose as the path of integration the line from $\infty$ to $P(x, y, z)$ that, extended, meets $q$. Along this path,
\[ ds = -dr' \quad \text{and} \quad E = -E' = -\frac{q}{4\pi \varepsilon_0 r'^2} \]

Consequently,
\[ \phi = -\int_{\infty}^{r} E \, ds = -\int_{\infty}^{r} \frac{q \, dr'}{4\pi \varepsilon_0 r'^2} = -\frac{q}{4\pi \varepsilon_0} \int_{r}^{\infty} \frac{dr'}{r'^2} = \frac{q}{4\pi \varepsilon_0 r} \]

26.6 What is an equipotential surface?

A surface (or, in two dimensions, a curve) on which the potential is constant is called an equipotential surface (curve). Thus, in cartesian coordinates, the equipotential surfaces are given by the equation $\phi(x, y, z) = c$, with one surface for each value of the constant $c$. In terms of the equipotentials, the relation between $\phi$ and $E$ may be geometrically expressed as follows: the field lines and the equipotentials are everywhere perpendicular. Indeed, for an infinitesimal displacement $ds$ lying in an equipotential surface, $d\phi = 0 = E \cdot ds$; and so $E$ must be perpendicular to $ds$.

For conductors in electrostatic equilibrium (no moving charges) the surfaces are equipotential surfaces. (Indeed, since the electric field is zero inside the conductor, the entire conductor is at constant potential.)
26.7 Define the electronvolt.

The work done in carrying a charge +e (coulombs) through a potential rise of 1 V is defined to be 1 electronvolt (eV). Therefore,

\[ 1 \text{ eV} = (1.602 \times 10^{-19} \text{ C})(1 \text{ V}) = 1.602 \times 10^{-19} \text{ J} \]

Equivalently,

work or energy (in eV) = \frac{\text{work (in J)}}{\text{e}}

26.8 If in Fig. 26-1, \( q = 40 \mu \text{C}, x = 0.5 \text{ m}, y = 0.8 \text{ m}, z = 0.6 \text{ m}, \) what is the potential at point \( P? \)

Here \( r = (0.5^2 + 0.8^2 + 0.6^2)^{1/2} = 1.118 \text{ m}. \) Hence,

\[ \phi = k \frac{q}{r} = (9 \times 10^9) \frac{40 \times 10^{-6}}{1.118} = 3.22 \times 10^6 \text{ V} = 322 \text{ kV} \]

26.9 In Prob. 26.8, a charge \( q_1 = -9 \mu \text{C} \) is placed at \( P. \) Compute its electric potential energy.

\[ U = q_1 \phi = (9 \times 10^{-9})(3.22 \times 10^6) = 2.9 \text{ J} \]

26.10 A point charge, \( q_1 = +2 \mu \text{C}, \) is placed at the origin of coordinates. A second, \( q_2 = -3 \mu \text{C}, \) is placed on the \( x \) axis at \( x = 100 \text{ cm}. \) At what point (or points) on the \( x \) axis will the absolute potential be zero?

![Fig. 26-2](image)

The two charges are as shown in Fig. 26-2. At a point \( x \) along the \( x \) axis the potential is

\[ V(x) = k \left( \frac{q_1}{|x|} + \frac{q_2}{|x - 1|} \right) \]

Setting \( V = 0 \) yields

\[ \frac{2}{|x|} = \frac{3}{|x - 1|} = 0 \text{ or } 2|x| = 3|x - 1| \]

We consider three cases \( x > 1, 0 < x < 1, \) and \( x < 0. \)

For \( x > 1: \) \( 2(x - 1) = 3x \Rightarrow x = -2 \) \text{ (contradiction and no solution)}

For \( 0 < x < 1: \) \( 2(1 - x) = 3x \Rightarrow x = 0.4 \text{ or } x = 40 \text{ cm} \)

For \( x < 0: \) \( 2(1 - x) = -3x \Rightarrow x = -2 \text{ or } x = -200 \text{ cm} \)

26.11 Determine the absolute potential in air at a distance of 3 cm from a point charge of 50 nC.

The dielectric constant of air can be taken as 1.0. Then, \( V = \frac{1}{(4\pi \epsilon_0)}(q/r) = (9 \times 10^9)(5 \times 10^{-9})/(0.03) = 15 \text{ kV}. \)

26.12 Show that the absolute potential at point \( P \) in Fig. 26-3 is zero.

![Fig. 26-3](image)
\[
\phi_r = \frac{1}{4\pi\varepsilon_0} \frac{+q}{\sqrt{a^2 + b^2}} + \frac{1}{4\pi\varepsilon_0} \frac{-q}{\sqrt{a^2 + b^2}} = 0
\]

[Note how much simpler it is directly to add potentials (scalars) than it is to superpose electric fields (vectors) and then to integrate.]

**26.13** What is the absolute potential of a 25-cm-radius metal sphere that carries a charge of 65 nC?

The metal sphere is an equipotential surface and, by symmetry, the charge distributes itself uniformly on the surface. The potential is thus that of a point charge at the center:

\[ V = \frac{(9.0 \times 10^9)(65 \times 10^{-9})}{0.25} = +2340 \text{ V} \]

**26.14** A hollow sphere has a radius of 10 cm and bears a charge of 5 nC. If the sphere is isolated, what is the potential a distance of 50 cm from the center of the sphere?

As in Prob. 26.13, \( V = \frac{kQ}{r} = \frac{(9 \times 10^9)(5 \times 10^{-9})}{0.50} = 90 \text{ V} \)

**26.15** What is the potential at the point midway between charges of +2 and +5 \( \mu \)C which are 6 m apart?

From Fig. 26-4,

\[ V = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} = \frac{(9 \times 10^9)(2 \times 10^{-6})}{3} + \frac{(9 \times 10^9)(5 \times 10^{-6})}{3} = 21 \text{ kV} \]

**26.16** Three equal charges of +6 nC are located at the corners of an equilateral triangle whose sides are 12 cm long (Fig. 26-5). Find the potential at the center of the base of the triangle.

\[ V_p = (9 \times 10^9) \left( \frac{6 \times 10^{-9}}{0.06} + \frac{6 \times 10^{-9}}{0.06} + \frac{6 \times 10^{-9}}{0.104} \right) = 2320 \text{ V} \]

**26.17** Three charges are placed at three corners of a square (Fig. 26-6). Find the potential at point A.

**Fig. 26-4**

**Fig. 26-5**

**Fig. 26-6**
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\[ q_1 = 3.00 \times 10^{-9} \text{ C} \]
\[ q_2 = -2.00 \times 10^{-9} \text{ C} \]
\[ p \]
\[ 0.300 \text{ m} \]
\[ 0.200 \text{ m} \]
\[ 0.200 \text{ m} \]
\[ 0.200 \text{ m} \]

**Fig. 26-7**

\[ V_a = (9 \times 10^9 \left( \frac{4 \times 10^{-9}}{0.20} + \frac{-4 \times 10^{-9}}{0.283} + \frac{4 \times 10^{-9}}{0.20} \right) = 233 \text{ V} \]

**26.18** Figure 26-7 shows a triangular array of three point charges. Find the electric potential \( V \) of these source charges at the midpoint \( P \) of the base of the triangle.

\[ V = (8.99 \times 10^9 \text{ N m}^{-2} \text{ C}^{-1}) \left( \frac{+1.00 \times 10^{-9} \text{ C}}{0.200 \text{ m}} + \frac{-2.00 \times 10^{-9} \text{ C}}{0.200 \text{ m}} + \frac{+3.00 \times 10^{-9} \text{ C}}{0.300 \text{ m}} \right) = 45.9 \text{ kV} \]

**26.19** A metal sphere 30 cm in radius is positively charged with 2 nC. Find the potential at the center of the sphere, on the sphere, and at 1 m from the center of the sphere.

The potential of the sphere or of any point inside is the same (see Prob. 26.6).

\[ V = kQ = \frac{(9 \times 10^9)(2 \times 10^{-9})}{0.30} = 60 \text{ kV} \] on the surface or at the center

For an external point, all charge may be considered concentrated at the center of the sphere.

\[ V = \frac{kQ}{r} = \frac{(9 \times 10^9)(2 \times 10^{-9})}{1} = 18 \text{ kV} \] at 1 m from the center

**26.20** Two metal spheres (radii \( a \) and \( b \)) are very far apart but are connected by a thin wire. Their combined charge is \( Q \). (a) What is the charge on each? (b) What is their absolute potential?

(a) Because a wire connects them, both spheres must be at the same potential. From Prob. 26.13, \( V_a = (kQ)/a = V_b = (kQ)/b \). Also \( Q = Q_a + Q_b \); so solving between these relations, \( Q_a = (Q)/(a + b) \) and \( Q_b = (Q)/(a + b) \). (b) Substitute into either \( V_a \) or \( V_b \) to find \( V = (kQ)/(a + b) \).

**26.21** Find the potential due to a thin uniformly charged rod of length \( 2a \).

(a) Take the rod along the \( Z \) axis, as in Fig. 26-8. By the rotational symmetry of the problem, it is enough to
determine the potential at an arbitrary point of the YZ plane. At the point \((y, z)\), the potential due to the element of charge \(dq = \lambda \, ds\) is

\[
d\phi = \frac{dq}{4\pi \varepsilon_0} = \frac{\lambda \, ds}{4\pi \varepsilon_0 \sqrt{y^2 + (z - s)^2}}
\]

Hence,

\[
\phi(y, z) = \frac{1}{4\pi \varepsilon_0} \int_{-\infty}^{\infty} \frac{\lambda \, ds}{\sqrt{y^2 + (z - s)^2}} = \frac{\lambda}{4\pi \varepsilon_0} \int_{-\infty}^{\infty} \frac{ds}{\sqrt{s^2 - 2as + y^2 + z^2}} - \frac{\lambda}{4\pi \varepsilon_0} \ln \left[\frac{y^2 + (z - a)^2}{y^2 + (z + a)^2}\right] - (z - a)
\]

The potential at point \((x, y, z)\) is obtained by replacing \(y^2\) by \(x^2 + y^2\) in the above expression.

26.22 In Prob. 26.21, set \(\lambda = q / 2a\) and show that as \(a \to 0\) the potential due to a point charge \(q\) located at the origin is obtained.

\[
\phi(x, y, z) = \frac{q}{4\pi \varepsilon_0} \left( \frac{1}{2a} \int_{-\infty}^{\infty} \frac{1}{\sqrt{y^2 + (z - s)^2}} \, ds \right)
\]

The expression in brackets is just the mean value of \(\frac{y^2 + (z - s)^2}{\sqrt{y^2 + (z - s)^2}}\) over the interval \(-a < s < a\); hence, as \(a \to 0\), \(\{\} \to \left[ \frac{y^2 + (z - 0)^2}{\sqrt{y^2 + (z - 0)^2}} \right] = \frac{1}{r}\), and so \(\phi(x, y, z) \to q / (4\pi \varepsilon_0 r)\). [The result may also be obtained from the logarithmic expression for \(\phi\).]

![Fig. 26-9](image)

26.23 Refer to Fig. 26-9 and write an expression for the potential \(\phi(x, y)\) at a general point \(P(x, y)\) in the XY plane.

Since \(r_1 = (x^2 + y^2)^{1/2}\) and \(r_2 = [(l - x)^2 + y^2]^{1/2}\),

\[
\phi(x, y) = \frac{1}{4\pi \varepsilon_0} \left( \frac{q_1}{r_1} + \frac{q_2}{r_2} \right) = \frac{1}{4\pi \varepsilon_0} \left[ \frac{q_1}{(x^2 + y^2)^{1/2}} + \frac{q_2}{(x^2 + y^2 + l^2 - 2xl)^{1/2}} \right]
\]

26.24 In the case \(q_1 = +|q|\) and \(q_2 = -|q|\), the array of Fig. 26-9 constitutes an electric dipole (of moment \(\mu = |q|l\)). Find the electric potential of the dipole at a point \(P\) whose distance \(r\) from the center of the dipole is large compared with \(l\).

The law of cosines gives

\[
r_1^2 = r^2 + \left( \frac{l}{2} \right)^2 - rl \cos \theta \\
r_2^2 = r^2 + \left( \frac{l}{2} \right)^2 + rl \cos \theta
\]

so that \(r_2^2 - r_1^2 = 2rl \cos \theta\). Hence,

\[
\phi = \frac{|q|}{4\pi \varepsilon_0} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{|q|}{4\pi \varepsilon_0} \frac{r_2 - r_1}{r_1 r_2 (r_1 + r_2)} = \frac{|q|}{4\pi \varepsilon_0} \frac{2rl \cos \theta}{r_2 (r_1 + r_2)}
\]
But, when \( r \gg l \), we have \( r_2r_1 = r^2 \) and \( r_2 + r_1 = 2r \), giving

\[
\phi \approx \frac{|q|}{4\pi \varepsilon_0} \frac{2rl \cos \theta - \mu \cos \theta}{2r^2} \frac{\mu \cos \theta}{4\pi \varepsilon_0 r^2} \]

This approximate formula becomes exact when \( l \to 0 \) and \( |q| \to \infty \) such that \( \mu \) remains constant and the two charges form a point dipole.

### 26.2 THE POTENTIAL FUNCTION AND THE ASSOCIATED ELECTRIC FIELD

#### 26.25

A circular disk carries a surface charge \( \sigma \text{ (C/m}^2 \text{)} \). Show that the electric field at any point on the axis of the disk depends only on \( \sigma \) and the angle \( \alpha \) subtended there by the disk.

\[ \text{Fig. 26-10} \]

We consider the charged disk of radius \( R \) to lie in the \( xy \) plane centered at the origin (Fig. 26-10). Rather than calculate the field along the \( z \) axis directly, we first calculate the potential, \( V(z) \). We consider the disk to be made up of concentric rings, each ring contributing to the potential:

\[
dV = \frac{\sigma}{2\varepsilon_0} \left( \frac{2\pi r dr}{r^2 + z^2} \right) \frac{1}{r^{1/2}}
\]

We then integrate from \( r = 0 \) to \( r = R \):

\[
V(z) = \frac{\sigma}{2\varepsilon_0} \int_0^R \left( \frac{r dr}{r^2 + z^2} \right)^{1/2} = \frac{\sigma}{2\varepsilon_0} \frac{1}{2} \left( \frac{r^2 + z^2}{r^2} \right)^{1/2} \bigg|_0^R = \frac{\sigma}{2\varepsilon_0} \left( \frac{R^2 + z^2}{R^2} - |z| \right)
\]

(1)

Noting that \( |z|/(R^2 + z^2)^{1/2} = \cos(\alpha/2) \) we have, factoring \( |z| \) out of the brackets,

\[
V(z) = \frac{\sigma |z|}{2\varepsilon_0} \left( \frac{1}{\cos(\alpha/2)} - 1 \right)
\]

Noting that by symmetry the electric field along the \( z \) axis is parallel to that axis, we have, using form (1),

\[
E(z) = E_z(z)k,
\]

with

\[
E_z(z) = -\frac{dV(z)}{dz} = -V'(z) = \begin{cases} 
\frac{\sigma}{2\varepsilon_0} \left( 1 - \frac{z}{(R^2 + z^2)^{1/2}} \right) & z > 0 \\
-\frac{\sigma}{2\varepsilon_0} \left( 1 + \frac{z}{(R^2 + z^2)^{1/2}} \right) & z < 0
\end{cases}
\]

\[
= \begin{cases} 
\frac{\sigma}{2\varepsilon_0} \left( 1 - \cos \frac{\alpha}{2} \right) & z > 0 \\
-\frac{\sigma}{2\varepsilon_0} \left( 1 - \cos \frac{\alpha}{2} \right) & z < 0
\end{cases}
\]

#### 26.26

Show that if the disk of Prob. 26.25 has a fixed charge, \( q \), the potential on the \( z \) axis reduces to that of a point charge at the origin in the limit \( R/|z| \to 0 \) (i.e., far out on the \( z \) axis).

By (1) of Prob. 26.25,

\[
V(z) = \frac{\sigma}{2\varepsilon_0} \left( (R^2 + z^2)^{1/2} - |z| \right)
\]

We note that \( \sigma - q/(\pi R^2) \) and that \((R^2 + z^2)^{1/2} = |z| (1 + R^2/z^2)^{1/2} \sim |z| (1 + 1/(R^2/z^2))\), for \( R/|z| \ll 1 \). Then,

\[
V(z) \sim \frac{q|z|}{2\pi \varepsilon_0 R^2} \frac{1}{2z^2} = \frac{1}{4\pi \varepsilon_0 |z|}
\]
(In general, any finite distribution of charge "looks like" a single point charge from a sufficient distance.)

26.27 If the potential in the region of space near the point (−2, 4, 6 m) is \( V = 80x^2 + 60y^2 \) V, what are the three components of the electric field at that point?

I We have \( E_x = -\partial V/\partial x = -160x = 320 \) V/m, and \( E_y = -\partial V/\partial y = -480 \) V/m. Similarly, \( E_z = 0 \).

26.28 The electric field \( \vec{E} \) set up by a certain distribution of charges is two-dimensional, with \( X \) component

\[
E_x = A[(x - l)(x^2 + y^2 + l^2) - x(x^2 + y^2)]^{1/2}
\]

where \( A \) and \( l \) are constants. Determine \( E_y \).

I We know that \( E_y = -\partial \phi/\partial y \). To obtain \( \phi \) we note that

\[
\phi(x, y) = -\int_{x'}^{x} E_x \, dx = - A \left( \int_{x'}^{x} \frac{\lvert x - l \rvert \, dx}{[x^2 + y^2]^{1/2}} - \int_{x}^{x'} \frac{x \, dx}{[x^2 + y^2]^{1/2}} \right)
\]

Noting that \( x \, dx = \lvert d(x^2) \rvert/2 \) and \( (x - l) \, dx = \lvert d(u^2) \rvert/2 \), with \( u = x - l \), we get \( \phi(x, y) = A[(x^2 + y^2 + l^2 - 2lx) - (x^2 + y^2)]^{1/2} \). Then \( E_x = A[(x^2 + y^2 + l^2 - 2lx) - (x^2 + y^2)]^{1/2} \).

26.29 The electric field outside a charged long straight wire is given by \( \vec{E} = -\frac{kQ}{r} \) V/m and is radially inward. What is the sign of the charge on the wire? Find the value of \( V_B - V_A \) if \( r_B = 60 \) cm and \( r_A = 30 \) cm. Which point is at the higher potential?

I The field is directed toward the wire and so the wire is charged negatively. Going from \( A \) to \( B \) is opposite to the direction of the field, so \( B \) is at a higher potential than \( A \). Thus

\[
V_B - V_A = -\int_{r_A}^{r_B} \frac{kQ}{r} \, dr = 5000 \ln 2 = 3470 \text{ V}
\]

26.30 A 30-cm-diameter metal sphere hangs from a thread in a very large room, so its surroundings are essentially at infinity. If the electric field at its surface is to be equal to the breakdown strength of air, 3 MV/m, what must be the absolute potential of the sphere?

I \( E \) just outside the surface is 3 \( \times \) \( 10^6 \) V/m; this is due to the charge \( Q \) on the uniformly charged sphere. Now \( E = (kQ)/r^2 \), while \( V \) just at the surface is \( V = (kQ)/r \). Taking ratios, \( V = ER = (3 \times 10^6)(0.15) = 4.5 \times 10^5 \) V. No field exists in the metal, so \( V = 450 \) kV throughout the volume of the metal.

26.31 The potential at a point \( P \) a distance \( y \) above the bottom negative plate of a parallel-plate combination is \( V(y) - V(0) = ky \). In this expression \( V(0) \) is the potential at the lower plate, and \( V(y) \) is the potential at the height \( y \) above it. Find \( E \) between the parallel plates.

I The situation can be simplified if we take \( V(0) = 0 \). This is allowable since the zero potential is arbitrary. We then have \( V(y) = ky \). Making use of results of Prob. 26.3 we have

\[
E_x = -\frac{\partial}{\partial x} (ky) = 0 \quad E_y = -\frac{\partial}{\partial y} (ky) = -k \quad E_z = -\frac{\partial}{\partial z} (ky) = 0
\]

The field therefore points downward (in the \( -y \) direction) between the plates and has a constant value \( k \), as expected.

26.32 Suppose the parallel metal plates shown in Fig. 26-11 are spaced 0.50 cm apart and are connected to a 90-V battery. Find the electric field between them and the surface charge density on the plates

![Fig. 26-11](image)

I We know that \( V_B - V_A = 90 \) V and that \( E \) between the plates is constant. Thus

\[
V_B - V_A = Ed \quad \text{or} \quad 90 = E(5 \times 10^{-1} \text{ m})
\]

Solving gives \( E = 18 \text{ kV/m} \). By Prob. 25.71, the charge density on the positive plate is \( \sigma = \varepsilon_0 E = (8.85 \times 10^{-12} \text{ C}^2/\text{N \cdot m}^2)(18000 \text{ N/C}) = 159 \text{ nC/m}^2 \); similarly, the value of \( \sigma \) at the negative plate is \( -159 \text{ nC/m}^2 \).
26.33 A potential difference of 150 V is applied to two parallel metal plates. If an electric field of 5000 V/m is produced between the plates, how far apart are the plates?

\[ E = \frac{V}{d} \quad \text{or} \quad 5000 = \frac{150}{d} \quad \text{and} \quad d = 0.03 \text{ m} = 3 \text{ cm} \]

26.34 The charge on an electron is \( 1.6 \times 10^{-19} \) C in magnitude. An oil drop has a weight of \( 3.2 \times 10^{-13} \) N. With an electric field of \( 5 \times 10^5 \) V/m between the plates of Millikan's oil-drop apparatus, this drop is observed to be essentially balanced. What is the charge on the drop in electronic charge units?

\[ E = 5 \times 10^5 \text{ V/m} = 5 \times 10^4 \text{ N/C}. \text{ For equilibrium, } F = Eq = mg \]

\[ 5 \times 10^5 q = 3.2 \times 10^{-13} \quad q = 6.4 \times 10^{-19} \text{ C} \quad \text{on the oil drop} \]

\[ \frac{q}{e} = \frac{6.4 \times 10^{-19}}{1.6 \times 10^{-19}} = 4 \text{ electrons} \]

26.35 In the Millikan experiment, an oil drop carries four electronic charges and has a mass of \( 1.8 \times 10^{-12} \) g. It is held almost at rest between two horizontal charged plates 1.8 cm apart. What voltage must there be between the two charged plates?

\[ E = V/d. \text{ Set the upward electric force equal to the downward force of gravity.} \]

\[ F = Eq = mg \quad \frac{Vq}{d} = mg \quad \frac{V(4 \times 1.60 \times 10^{-19})}{0.018} = (1.8 \times 10^{-12})(9.8) \quad \text{and} \quad V = 496 \text{ V} \]

26.36 Two large parallel metal plates (3.00 mm apart) are charged to a potential difference of 12 V. (a) What is the field between them? (b) They are now disconnected from the battery and pulled apart to 5.00 mm. What is the new electric field between them and what is now the potential difference?

\[ (a) \text{ From } V = Ed \text{ we have } E = 12.0/3.00 \times 10^{-2} = 4000 \text{ V/m}. \text{ (b) Since the battery was removed, the charge remains the same, as does } E (E = \sigma/e_0). \text{ Potential } = 4000(5.00 \times 10^{-5}) = 200 \text{ V} \]

26.37 Two very large flat metal plates are parallel and separated by a distance \( D \). The side of the left plate that faces the right plate has a surface charge \( +\sigma \). (a) What is the field between the plates? (b) The potential difference? (c) Another metal plate, uncharged, is placed between these two without altering the charge on the original plates. Its thickness is \( d < D \). What is the field in the gap between it and the left plate? (d) In the other gap? (e) What is now the potential difference between the two outer plates?

![Figure 26-12](image)

![Fig. 26-12](image)

\[ E = \sigma_0 \text{ with } E = \sigma_0 \]

26.38 A pair of horizontal metal plates are separated by a vertical distance of 10 cm, and the voltage different between them is 28 V. A small ball of 0.60-g mass hangs by a thread from the upper plate. What is the tension in the thread if the ball carries a charge of 20 \( \mu \text{C} \)? Two answers are possible. Find both.

\[ \text{We use the condition for equilibrium: sum of forces equals zero. Tension } T \text{ on the mass is up, weight } mg \]

\[ T = mg \text{ up, } F = Eq \text{ horizontally. } \]
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is down, and \( qE \) can be up or down. We have \( mg = (6.0 \times 10^{-3})(9.8) = 5.88 \text{ mN} \), \( qE = (qV)/d = [(20 \mu \text{C})(28 \text{ V})]/(0.10 \text{ m}) = 5.60 \text{ mN} \), so that \( T = mg + qE = 11.5 \text{ MN} \) or \( 0.28 \text{ MN} \).

26.39 The electric field in a certain region is given by \( E = 3\hat{i} - 3\hat{j} \text{ kV/m} \). Find the difference in potential \( V_B - V_A \) if \( A \) is at the coordinate origin and point \( B \) is (a) \((0, 0, 5) \text{ m}\), (b) \((4, 0, 3) \text{ m}\).

Because the field is conservative, we can use straight-line paths parallel to the coordinate axes (see Fig. 26-13). (a) No work is done in carrying a unit positive charge, since there is no field component in the \( z \) direction. There is zero potential difference between the origin and \((0, 0, 5) \). (b) Following the path from \((0, 0, 0) \) to \((4, 0, 0) \) to \((4, 0, 3) \), the work done against the field in carrying a unit positive charge (the potential difference) is \( -E_x(4 - 0) - E_y(3 - 0) = -20 \text{ kV} \).

[Diagram: Fig. 26-13]

26.40 For the field given in Prob. 26.39, find the potential difference \( V_B - V_A \) between the two points \( A = (0, 5, -1) \) and \( B = (-3, 2, 2) \text{ m} \). Repeat for the points \( A = (0, 5, -1) \) and \( B = (3, 2, 2) \text{ m} \).

Take as the path the broken line from \((0, 5, -1) \) to \((-3, 5, -1) \) to \((-3, 2, -1) \) to \((-3, 2, 2) \). We have \( V_B - V_A = -(5000)(-3 - 0) - (-3000)(2 - 5) - (0)(2 - (-1)) = 6 \text{ kV} \). In the second case the \( x \) calculation is changed to \(-(5000)(3 - 0)\), so the answer becomes \(-24 \text{ kV} \).

26.41 In a certain region of space, the electric field is directed in the +\( y \) direction and has a magnitude of 4000 \text{ V/m} \. What is the potential difference from the coordinate origin to the following points? (a) \( x = 0, y = 20 \text{ cm}, z = 0 \); (b) \( x = 0, y = -30 \text{ cm}, z = 0 \); (c) \( x = 0, y = 0, z = 15 \text{ cm} \).

In each case work is done only in moving along the \( y \) axis: \( V - V_0 = -E_y(y) \). (a) \( V - V_0 = -(4000 \text{ V/m})(0.20 \text{ m}) = -800 \text{ V} \). (b) \( V - V_0 = -(4000 \text{ V/m})(-0.30 \text{ m}) = 1200 \text{ V} \). (c) \( V - V_0 = -(4000 \text{ V/m})(0) = 0 \text{ V} \).

26.42 Use \( \Delta V = -E \Delta s \) to compute \( V_{AB}, V_{BC}, \) and \( V_{CA} \) in Fig. 26-14. Using these results, show that the work required to carry a charge \( q \) from \( A \) to \( B \) to \( C \) and back to \( A \) is zero.

[Diagram: Fig. 26-14]
ELECTRIC POTENTIAL AND CAPACITANCE

26.43 Give an informal proof of the following statement: The electric potential cannot assume a minimum (or a maximum) value inside a charge-free region.

Suppose, on the contrary that the potential took on a local minimum value, \( \phi_0 \), at the interior point \( P_0 \). Then (Fig. 26-15) we could enclose \( P_0 \) in a gaussian sphere of so small a radius, \( a \), that (i) the sphere lies entirely within the charge-free region, and (ii) \( \phi = \phi_0 \) at every point of the spherical surface. Gauss' law, applied to this sphere, would give

\[
0 = \int_S \mathbf{E} \cdot d\mathbf{S} = \int_S E_n dS
\]

where \( E_n \) is the normal (radial) component of the field at the surface of the sphere. But, by definition of the electric potential,

\[
E_n = -\frac{d\phi}{dn}
\]

that is, the derivative of the potential in the radial direction. Now, by decreasing \( a \) if necessary, we can preserve conditions (i) and (ii) above and at the same time allow the derivative in (2) to be approximated by a difference-quotient to any required degree of accuracy. Thus,

\[
E_n = \frac{\Delta \phi}{\Delta n} = -\frac{\phi - \phi_0}{a}
\]

and (1) becomes

\[
0 = \int_S (\phi - \phi_0) dS
\]

But (4) is impossible: \( \phi - \phi_0 \) is nonnegative at each point of \( S \) and so its integral over \( S \) must be positive. This contradiction establishes the desired result.

The implication is very significant: No charge placed in an electrostatic field can be in stable equilibrium, since that requires being at a minimum of potential energy. Note that unstable equilibrium (see, e.g., Prob. 25.21) does not demand a potential energy maximum, but only a saddle point.

26.3 ENERGETICS; PROBLEMS WITH MOVING CHARGES

26.44 It requires 50 \( \mu J \) of work to carry a 2-\( \mu C \) charge from point \( R \) to \( S \). What is the potential difference between the points? Which point is at the higher potential?

\[
V_S - V_R = \frac{W}{q} = \frac{5 \times 10^{-4}}{2 \times 10^{-12}} = 25 \text{ V}
\]

Point \( S \) is at the higher potential since work is needed to carry a positive charge from \( R \) to \( S \).

26.45 The electron in a hydrogen atom is most probably at a distance \( r = 5.29 \times 10^{-11} \text{ m} \) from the proton, which is the nucleus of the atom. Evaluate the electric potential energy \( U \) of the atom.

The proton is so much more massive than the electron that we identify \( U \) with the work needed to bring
the electron from $\infty$ to point $r$ in the field of the stationary proton. The potential established by the proton at distance $r$ is:

$$V = \frac{1}{4\pi \varepsilon_0} \frac{e}{r} = \frac{1}{4\pi \varepsilon_0} \frac{8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \times 1.60 \times 10^{-19} \text{ C}}{5.29 \times 10^{-11} \text{ m}} = 27.2 \text{ V}$$

Hence, $U = -eV = -27.2 \text{ eV}$. The Coulomb force is attractive, giving the atom a negative potential energy relative to infinity.

26.46 Compute the electric potential energy, $U$, for the charge configuration shown in Fig. 26-16.

- We bring the three charges to their locations, from $\infty$, in succession. No work is needed to bring up the first charge; take it to be the $5 \mu C$. The work needed to bring up the $-2 \mu C$ from $\infty$ is $(-2 \mu C)(5 \mu C)(9 \times 10^9)/0.20 = -0.45 \text{ J}$. These two charges set up a potential at the location of the third charge of $V = (9 \times 10^9)/0.20(5 \mu C - 2 \mu C) = +1.35 \times 10^5 \text{ V}$. To bring up the $-3 \mu C$ charge therefore requires $-0.405 \text{ J}$. The total work needed is $-0.855 \text{ J}$.

Note that this is just the sum of the three pairwise potential energies, each calculated as if it were an isolated pair. It is not, however, the sum $q_1V_1 + q_2V_2 + q_3V_3$, where $V_1, V_2, V_3$ are the potentials in each case due to the other two charges. This latter sum double-counts the energy.

26.47 Problem 26.46 suggests the formula

$$U = \frac{1}{2}(q_1V_1 + q_2V_2 + \cdots + q_nV_n)$$

for the potential energy of an arbitrary configuration of $n \geq 2$ point charges. Obtain this formula.

- Let us introduce the notation $V_i(i)$ for the potential at the location of $q_i$ due to the source charge $q_i$ from superposition,

$$V_i = V_i(1) + V_i(2) + \cdots + V_i(i-1) + V_i(i+1) + \cdots + V_i(n)$$

for $i = 1, 2, \ldots, n$. First suppose the charges to be individually brought in from infinity in the order $1, 2, \ldots, n$. The work required is

$$U = 0 + q_1V_1(1) + q_1[V_1(1) + V_1(2)] + \cdots + q_n[V_n(1) + V_n(2) + \cdots + V_n(n-1)]$$

Now bring in the charges in the order $n, n-1, \ldots, 1$,

$$U = 0 + q_{n+1}V_{n-1}(n) + q_{n+2}[V_{n-2}(n) + V_{n-1}(n-1)] + \cdots + q_n[V_1(n) + V_1(n-1) + \cdots + V_1(2)]$$

When (2) and (3) are added and divided by 2 and (1) is used to simplify the coefficient of $q_i$, the desired formula results.

For a continuous distribution of charge over a region $\mathcal{R}$, the calculus generalization of our formula is obviously

$$U = \frac{1}{2} \int_{\mathcal{R}} V \, dq = \frac{1}{2} \int_{\mathcal{R}} \rho \, d\mathcal{V}$$

where $\rho$ is the charge density and $d\mathcal{V} = dx \, dy \, dz$ the element of volume.

26.48 Find the electric potential energy of an isolated metal sphere of radius $R$ with total charge $Q$.

- Recalling that the spherical surface is an equipotential, with $V = Q/(4\pi \varepsilon_0 R)$, we have from (4) of Prob. 26.47,

$$U = \frac{1}{2} \int_{\text{surface}} V \, dq = \frac{1}{2} Q \frac{V}{Q} = \frac{Q^2}{8\pi \varepsilon_0 R}$$

More directly, we can suppose the charge brought in from infinity in tiny increments $dq$. If charge $q$ has already been assembled, the work needed to bring in the next $dq$ is $dU = V(q) \, dq = (q/4\pi \varepsilon_0 R) \, dq$. Integrating from $q = 0$ to $q = Q$ gives $U = Q^2/(8\pi \varepsilon_0 R)$, as before.

26.49 In Fig. 26-17, a sphere carrying a uniformly distributed charge $Q = 40 \mu C$ is located at the origin: $r_1 = 0.5 \text{ m}$ and $r_2 = 1.2 \text{ m}$. (a) Find the potential at $P_1$ and at $P_2$ due to $Q$. What is the potential at $P_1$ with respect to $P_2$?
(b) A small sphere carrying charge \( q = 8 \, \mu\text{C} \) is placed at \( P_1 \). What is its potential energy (relative to \( \infty \))?
(c) Suppose that the small sphere moves freely from \( P_1 \) through \( P_2 \). In the trip from \( P_1 \) to \( P_2 \), what is its change in kinetic energy?

The potential at \( P_1 \) with respect to \( P_2 \) is the absolute potential at \( P_1 \) minus the absolute potential at \( P_2 \). We denote this relative potential as \( \phi_{12} \) or \( V_{12} \). Then

\[
\phi_{12} = \phi_1 - \phi_2 = 720 - 300 = 420 \, \text{kV}
\]

(b) \( U_l = q\phi_1 = (8 \times 10^{-6})(720 \times 10^3) = 5.76 \, \text{J} \)

(c) \( K_2 - K_1 = U_1 - U_2 = q(\phi_1 - \phi_2) = q\phi_{12} = (8 \times 10^{-6})(420 \times 10^3) = 3.36 \, \text{J} \)

The difference in potential, \( V_{AB} = V_A - V_B \), between wires \( A \) and \( B \), Fig. 26-18, as measured by a voltmeter is 6000 V. A small sphere having mass 0.150 kg and carrying a charge \( q = +500 \, \mu\text{C} \) is released from rest at a point very near wire \( A \) and allowed to move to wire \( B \). (a) Does \( V_{AB} \) depend on \( s \), the distance between the wires? (b) What work \( W_E \) is done by electric forces on the sphere? (c) With what speed \( u \) will the sphere arrive at \( B \)? (d) What is the average field \( E_{avg} \) between \( A \) and \( B \)?

(a) No.
(b) \( W_E = U_A - U_B = qV_{AB} = (500 \times 10^{-6})(6000) = 3.00 \, \text{J} \)
(c) \( W_T = W_E + W_G = \frac{1}{2}mu^2 \quad W_G = mgh = (0.150)(9.8)(0.2) = 0.29 \, \text{J} \)

Thus

\[
\frac{1}{2}mu^2 = 3.29 \, \text{J} \quad \text{and} \quad u = \sqrt{\frac{6.58}{0.15}} = 6.6 \, \text{m/s}
\]

(d) \( E_{avg} = \frac{V_{AB}}{s} = \frac{6000}{0.20} = 30 \, \text{kV/m} \)