Theorems about Power Series

Taylor series

The coefficients c_n of the power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

for the function $f(z)$ are given by:

$$c_n = \frac{1}{n!} f^{(n)}(z)|_{z=a} = \frac{1}{n!} \frac{d^n}{dz^n} f(z)|_{z=a}$$

Example:

$$\sin(z) = \sum_{n=0}^{\infty} c_n z^n$$

where

$$c_n = \frac{1}{n!} \frac{d^n}{dz^n} \sin(z)|_{z=0} = \begin{cases} 0 & n \text{ even} \\ \frac{(-1)^{(n-1)/2}}{n!} & n \text{ odd} \end{cases}$$

Convergence

If a function has a power series expansion around some point a, then the region in which the series accurately represents the function (i.e. it's circle of convergence) is shaped like a circle centered at a that extends to the nearest point at which the function is not analytic. (Analyticity is a technical term that you will learn about later. Briefly, a function that is not analytic at a point is singular in some way. A function is certainly not analytic at any point at which its value becomes infinite or at a branch point of a root.)

Uniqueness

The power series of a function, if it exits, is unique, i.e. there is at most one power series of the form $\sum_{n=0}^{\infty} c_n(z-a)^n$ which converges to a given function within a circle of convergence centered at a. We call this a power series "expanded around a".

Note: This theorem is an open invitation to collect a bag of cute tricks. It doesn't matter how you find a series for a function, once you have it, it is **the** series. The rest of these theorems should be in your bag of cute tricks.

1. A power series may be differentiated or integrated term-by-term. The resulting series converges to the derivative or integral of the function represented by the original series within the same circle of convergence as the original series.

   Example:
$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad \forall z$$
$$\cos z = \frac{d}{dz} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) z^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \qquad \forall z$$

2. One series may be substituted in another provided that the values of the substituted series are in the circle of convergence of the other series.

   Example:
$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots = \sum_{n=0}^{\infty} (-z)^n \qquad |z| < 1$$

$$\frac{1}{1+\sin z} = 1 - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \right)^2 - \left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \right)^3 + \ldots$$
$$= 1 - z + z^2 + (\frac{1}{3!} - 1)\, z^3 + \ldots \qquad |\sin(z)| < 1$$

   What happens if you try this same trick to find a power series for $1/(1+\cos z)$? Why?

   Another example:
$$z - \frac{\pi}{2} = z - \frac{\pi}{2} \qquad \forall z$$

   Note: This is a very short power series with just two non-zero terms.

$$\cos(z) = -\sin(z - \frac{\pi}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (z - \frac{\pi}{2})^{2n+1}}{(2n+1)!} \qquad \forall z$$

   Note: Starting with a power series for $\sin(z)$ expanded around $z = 0$, we have obtained a power series for $\cos(z)$ expanded around $z = \frac{\pi}{2}$.

3. Two power series of like powers may be added, subtracted, or multiplied. The resulting series converges at least within the common circle of convergence.

Example:
$$\begin{aligned}\frac{2}{1-z^2} &= \frac{1}{1+z} + \frac{1}{1-z}\\ &= (1 - z + z^2 - z^3 + \ldots) + (1 + z + z^2 + z^3 + \ldots)\\ &= 2(1 + z^2 + z^4 + \ldots) \qquad\qquad |z| < 1\end{aligned}$$

Compare this to the result you would get using the previous theorem. Which method is faster?

Another example:
$$\begin{aligned}\frac{\sin z}{1+z} &= (z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots)(1 - z + z^2 - z^3 + z^4 - z^5)\\ &= z - z^2 + (-\frac{1}{3!} + 1)z^3 + (\frac{1}{3!} - 1)z^4 + (\frac{1}{5!} - \frac{1}{3!} + 1)z^5 + \ldots \qquad\qquad |z| < 1\end{aligned}$$

Compare this series to the series for the function $1 - \frac{1}{1+\sin(z)}$ (see the first example in Theorem 2.) What can you conclude about the wisdom of assuming two series are the same if their first three terms are identical?

4. Two power series expanded around the same point may be divided. If the leading term(s) of the denominator series is not zero, or if the zero(s) is cancelled by the numerator, then the resulting series converges within some circle. If the radius of convergence of the numerator and denominator series are r_1 and r_2, respectively, and the distance from the origin of the circles to the nearest zero of the denominator series is s, then the quotient series converges at least inside the smallest of the three circles of radii r_1, r_2, and s.

Try the previous example $\sin z/(1 + z)$ using synthetic division, instead. Is this method easier or harder? Imagine what you would do if the denominator were a power series with an infinite number of non-zero terms.

5. The series expansions for most functions recorded in books are expansions around the point $z = 0$. To expand around a point $a \neq 0$ write every z which appears in the function as $(z - a) + a$, simplify creatively, and use Theorem 2.

Example: Expand $\sin z$ around $z = \pi$.
$$\begin{aligned}\sin z &= \sin[(z-\pi) + \pi]\\ &= \sin(z-\pi)\cos\pi + \cos(z-\pi)\sin\pi\\ &= -\sin(z-\pi)\\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n (z-\pi)^{2n+1}}{(2n+1)!} \qquad\qquad \forall z\end{aligned}$$

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