PH 451: Capstone in Quantum Mechanics

Homework 2

Due 1/16/09

1. Establish that the operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ is Hermitian. Recall that the definition of a Hermitian operator is one that is equal to its Hermitian conjugate and that the Hermitian conjugate is defined by $\langle\psi|\hat{A}^\dagger|\phi\rangle = \langle\phi|\hat{A}|\psi\rangle^*$. So your job is to find $\hat{p}^\dagger$, given the definition of *p* and the definition of the Hermitian conjugate, and show that it is equal to *p*. (Hint: Use integration by parts.)

2. Prove the important result that the Hermitian conjugate of a product of two operators is the product of the Hermitian conjugate of each, but in reverse order. Mathematically stated:

   $\left(\hat{A}\hat{B}\right)^\dagger = \hat{B}^\dagger\hat{A}^\dagger$. Note that the definition of the Hermitian conjugate used above can also be written as $\left\langle\hat{A}^\dagger\psi\middle|\phi\right\rangle = \left\langle\psi\middle|\hat{A}\phi\right\rangle = \langle\psi|\hat{A}|\phi\rangle$, or in words: "The operator $\hat{A}$ sandwiched between a bra and a ket can be considered as $\hat{A}$ acting forward on the ket, or as its conjugate $\hat{A}^\dagger$ acting backward on the bra." Use your result to decide whether the product operators (i) $\hat{x}\hat{p}$ and (ii) $\hat{x}\hat{p} + \hat{p}\hat{x}$ are Hermitian or not.

3. i) For the ground state $\psi_0(x)$ of the harmonic oscillator, calculate $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, and $\langle p^2\rangle$ by explicit spatial integration.

   ii) Calculate $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, and $\langle p^2\rangle$ for all energy eigenstates $|\psi_n\rangle$ of the harmonic oscillator without doing integration (*i.e.*, use the operators *a* and $a^\dagger$).

   iii) Check that the uncertainty principle is obeyed in the above cases.

4. For the ground state of the harmonic oscillator, calculate the probability that the particle is found in the classically forbidden region.