PH 422: Day 5

12 Curl

Consider a small rectangular loop in the $yz$-plane, with sides parallel to the coordinate axes. What is the circulation of $\vec{A}$ around this loop?

Consider first the horizontal edges, on each of which $d\vec{r} = dy \hat{j}$, although $dy < 0$ on top and $dy > 0$ on the bottom. Since we would like to compare these two edges, we will in this one instance assume that $dy > 0$, and insert an explicit minus sign on top. Thus,

$$\sum_{\text{top+bottom}} \vec{A} \cdot d\vec{r} = -\vec{A}(z + dz) \cdot \hat{j} dy + \vec{A}(z) \cdot \hat{j} dy$$

$$= -\left( A_y(z + dz) - A_y(z) \right) dy$$

$$= -\frac{A_y(z + dz) - A_y(z)}{dz} dy dz$$

$$= -\frac{\partial A_y}{\partial z} dy dz$$

Just as with the divergence, in making this argument we are assuming that $\vec{A}$ doesn’t change much in the $x$ and $y$ directions, while nonetheless caring about the change in the $z$ direction. We are again subtracting the values at two points separated in the $z$ direction, so we are canceling the zeroth order term in $z$, and therefore need the next order term. This can be made precise using a multivariable Taylor series expansion.

Repeating this argument for the remaining two sides leads to

$$\sum_{\text{sides}} \vec{A} \cdot d\vec{r} = \vec{A}(y + dy) \cdot \hat{k} dz - \vec{A}(y) \cdot \hat{k} dz$$

$$= \left( A_z(y + dy) - A_z(y) \right) dz$$

$$= \frac{A_z(y + dy) - A_z(y)}{dy} dy dz$$

$$= \frac{\partial A_z}{\partial y} dy dz$$

where care must be taken with the signs, which are different from the previous computation. Adding up both expressions, we obtain

$$\text{total } yz\text{-circulation} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) dx dy \quad (1)$$
Since this is proportional to the area of the loop, it approaches zero as the loop shrinks down to a point. The interesting quantity is therefore the ratio of the circulation to area. This is almost the curl.

We could have oriented our loop any way we liked. We have a circulation for each orientation, which we can associate with the normal vector to the loop; curl is a vector quantity. Our loop has counterclockwise orientation of the loop as seen from the positive $x$-axis; we are computing the $\hat{i}$-component of the curl.

Putting this all together, we have

$$\text{curl}(\vec{A}) \cdot \hat{i} = \frac{yz\text{-circulation}}{\text{unit area}} = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

(2)

The rectangular expression for the full curl now follows by cyclic symmetry, yielding

$$\text{curl}(\vec{A}) = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$$

(3)

which is more easily remembered in the form

$$\text{curl}(\vec{A}) = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

(4)

An analogous construction can be used in curvilinear coordinates; the results for spherical and cylindrical coordinates can be found on the inside front cover of Griffiths.

13 Stokes’ Theorem

The total circulation of the magnetic field around a small loop is given by

$$\text{circulation} = \sum_{\text{box}} \vec{B} \cdot d\vec{r} = (\vec{\nabla} \times \vec{B}) \cdot d\vec{A}$$

where $d\vec{A} = \hat{n} \, dA$, with $\hat{n}$ is perpendicular to the (filled in) loop. But any surface can be filled with such loops. Furthermore, the circulation around the edge of the surface is just the sum of the circulations around each of the smaller loops, since the circulation through any common side will be zero
(because adjacent loops have opposite notions of “around”). Thus, the total circulation around any loop is given by

\[ \oint_{\text{loop}} \vec{B} \cdot d\vec{r} = \int_{\text{inside}} (\vec{\nabla} \times \vec{B}) \cdot d\vec{A} \]

This is Stokes’ Theorem.