28  Differential form of Ampère’s Law

Recall that Ampère’s Law says that

$$\oint_{\text{loop}} \vec{B} \cdot d\vec{r} = \mu_0 I_{\text{inside}}$$

But the enclosed current is just

$$I_{\text{inside}} = \int_{\text{inside}} \vec{J} \cdot d\vec{A}$$

so we have

$$\oint_{\text{loop}} \vec{B} \cdot d\vec{r} = \mu_0 \int_{\text{inside}} \vec{J} \cdot d\vec{A}$$

Putting this all together, Stokes’ Theorem tells us that

$$\int_{\text{inside}} (\nabla \times \vec{B}) \cdot d\vec{A} = \mu_0 \int_{\text{inside}} \vec{J} \cdot d\vec{A}$$

for any closed loop. This means that the integrands themselves must be equal, that is,

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

This is the differential form of Ampère’s Law, and is one of Maxwell’s Equations. It states that the curl of the magnetic field at any point is just a measure of the current density there.

In the activity earlier this week, Ampère’s Law was used to derive the magnetic field for a symmetric current distribution. The differential form of Ampère’s Law makes it possible to go the other way.

29  Second derivatives and Maxwell’s Equations

The electric field is conservative, that is, it is the gradient of some function, namely the scalar potential:

$$\vec{E} = -\nabla V$$
But since
\[ \nabla \times \nabla V = \vec{0} \]
for any function \( V \), we can rewrite this as
\[ \nabla \times \vec{E} = \vec{0} \]
which is another of Maxwell’s Equations, namely the differential form of (the electrostatic version of) Faraday’s Law.

Similarly, we can apply the identity
\[ \nabla \cdot (\nabla \times \vec{F}) = 0 \]
for any vector field \( \vec{F} \), to the magnetic vector potential, which yields the fourth and final of Maxwell’s Equations, namely
\[ \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0 \]

In summary, Maxwell’s equations for electro- and magnetostatics are:
\[
\begin{align*}
\nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\
\nabla \times \vec{B} &= \mu_0 \vec{J} \\
\nabla \times \vec{E} &= \vec{0} \\
\nabla \cdot \vec{B} &= 0
\end{align*}
\]