Epistemic Games in Thermodynamics: A Cognitive Task Analysis

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Several studies in recent years have demonstrated that upper-division students struggle with the mathematics of thermodynamics. This paper presents a task analysis based on several expert attempts to solve a challenging mathematics problem in thermodynamics. The purpose of this paper is twofold. First, we highlight the importance of cognitive task analyses for understanding expert performance and show how the epistemic games framework can be used as a tool for this type of analysis, with thermodynamics as an example. Second, through this analysis, we identify several issues related to thermodynamics that are relevant to future research into student understanding and learning of the mathematics of thermodynamics.

I. INTRODUCTION

In a recent review of problem solving in physics, Maloney identified several open questions in the area of problem solving, including:

> "How do novices transition to expertise? What are the patterns in how peoples' domain knowledge, problem-solving skills, epistemology, etc. morph from novice status to expert status?"[1]

In order to address these questions, one must first establish what expertise looks like. However, as Maloney points out, a task is not inherently a problem — a problem exists only in the interaction between an individual and a task.

> "Different people interacting with the same task/situation might not all find it to be a problem. The skills and knowledge an individual brings to a situation play a major role in whether that individual thinks of a situation as a problem"[1]

There have been few studies that have looked at expert problem solving within the domain of physics and, most of those that do, focus on "end-of-chapter" problems in introductory physics which are typically not problems for experts. One of the few exceptions to this is Singh ,[2] where the given task, although it could be solved using introductory physics concepts and techniques, was nonintuitive and therefore posed a challenge to both physics students and professors.

One area of problem solving that is of particular interest to physics faculty is the use of mathematics and the connections between mathematics and physics. In particular, several studies in recent years demonstrate that students struggle to connect mathematics and physics.[3–5] In particular, students struggle with many mathematical aspects of thermodynamics[6–8] and with partial derivatives in particular.[9–12] With few exceptions,[13] research on student understanding of derivatives focuses on conceptual understanding of ordinary derivatives, particularly on rate of change and graphical understanding of derivatives.[14–17]

As a part of one author's (DR) recent work in redesigning the junior-level thermodynamics course at Oregon State,[18] three authors (CAM, TD, and DR) recently began to explore the possibility of using differentials within the context of thermodynamics. This was a logical extension of The Vector Calculus Bridge Project,[19] which found the vector differential to be a helpful bridge between vector calculus and upper-division electromagnetism courses and led to substantial reforms in how these courses are taught at Oregon State University.[20,21]

The current study grew out of these discussions and it is part of a larger project designed to better understand how and why students struggle with the mathematics of thermodynamics, how practicing physicists approach the connections between mathematics and physics in thermodynamics, and ultimately how to facilitate an appropriate transition from student to professional in this area.

The purpose of this paper is twofold. First, we highlight the importance of cognitive task analyses for understanding expert performance, particularly for tasks that can be validly addressed in different ways; and we use the epistemic games framework as a tool for this analysis. Second, through this analysis, we identify several issues related to expert use of partial derivatives in thermodynamics that are relevant to future research into student understanding and learning of the mathematics of thermodynamics.

In Section II, we present some background on the analytic frameworks that we are using. Following that, there is a brief discussion of the study design (Section III). Then we present descriptions of three epistemic games (Section IV) and two variations (Section V) that were observed in this study. Section VI compares the games presented here to each other and to previous games identified in the literature and discusses some of the interesting questions raised by this study. We then present some implications and limitations of this work, as well as the direction this work will go in the future (Section VII).

II. BACKGROUND

This section presents an overview of cognitive task analysis and epistemic games as the primary theoretical frameworks used in this study.

A. Cognitive Task Analysis

Research on expertise and expert performance has a long history and a rich array of methodologies available for analyzing expert performance,[22] including task analysis, which can be succinctly described as "the study of how work is achieved by tasks."[23]In *The Cambridge Handbook of Expertise and Expert Performance*,[22] Schraagen provides an excellent overview of the historical progression of task analysis and an introduction to some of the approaches within this field that focus on professional practitioners (*e.g.,* Hierarchical Task Analysis, Critical Incident Technique, GOMS model, *etc*).[24]

Given the cognitive nature of and importance of disciplinary knowledge for many of the tasks in physics, the emergence of *cognitive task analysis* (CTA), primarily within the Human-Computer Interaction community, is of particular relevance to the Physics Education Research community.

> "*Cognitive task analysis* is the extension of traditional task analysis techniques to yield information about the knowledge, thought processes, and goal structures that underlie observable task performance."[25]

There are many different methods that fall under the umbrella of CTA, and they typically focus on tasks that are highly discipline-specific and complex, something that is vital for understanding expertise in our context.

> "Task analysis now has a focus of understanding expert knowledge, reasoning, and performance, and leveraging that understanding into methods for training and decision support, to amplify and extend human abilities to know, perceive, and collaborate."[24]

This paper presents a cognitive task analysis for a mathematical task in thermodynamics at the upper-level undergraduate level. At the upper-level undergraduate level, students have already transitioned from novices to something more akin to journeymen.[26] At this level, even well-defined "end-of-chapter" problems have become more complex and there are often multiple ways to solve the same problem. To understand and describe problem solving that is correspondingly more complex and discipline specific requires leveraging the strengths of field of inquiry that is as rich and diverse as cognitive task analysis.

B. Epistemic Games

As mentioned above, there are many different methods for performing a cognitive task analysis. One that we have found particularly fruitful in this case is the framework of *epistemic games* (or e-games), originally proposed by Collins and Ferguson ,[27] who define an epistemic game as a complex "set of rules and strategies that guide inquiry." They use these games as a way to describe expert scientific behavior that could be considered normative within a given community. The use of e-games has since been extended in several ways, primarily as a descriptive analysis tool for understanding student behavior and classroom discourse.[28–34]

This paper proposes three new epistemic games and two variations as a means of providing a task analysis for a challenging thermodynamics problem. We extract the commonalities in the behavior of experts as a means of understanding ways of thinking that are reflective of the community. In this sense, we are consistent with Collins and Ferguson 's[27] use of epistemic games as depicting normative practices. However, we also draw these conclusions from the behavior of specific experts, similar to other studies that are more descriptive in nature (*e.g.,* Tuminaro and Redish [34]). Including both descriptive and normative elements allows for a discussion of what we currently and implicitly expect our students to learn and provokes questions such as whether the practices within the community serve the purposes we think they do and if so, whether the use of certain epistemic games should be more explicit in our instruction.

It is important to note that there are different perspectives on the specificity of epistemic games. Perkins [35] claims that epistemic games are a high-level, decontextualized form of reasoning and some of the epistemic games that have since been proposed are also in this camp (*e.g.,* the *Answer-Making Epistemic Game* proposed by Chen *et al.* [28]), while others are more discipline specific (*e.g.,* Lunk 's[29] *iterative-debugging* game). Most of the games proposed by Collins and Ferguson [27] are fairly discipline independent (*e.g., list game, compare-and-contrast game, cause-and-effect game, etc.*), but they also point out that

> "Different disciplines are characterized by the forms and games they use. As disciplines evolve, they develop more complex and more constrained epistemic forms and games. These are sometimes specialized to fit the subject matter being analyzed."

In addition, even Perkins allows that there are both general and specific forms of epistemic games. Given the goals of this project and the subject matter at hand, we believe a discipline specific perspective is not only appropriate but necessary. Thus, the games proposed in this paper are highly specialized and constrained.

Here we introduce the terminology and relevant features of epistemic games that we use throughout this pa-

TABLE I. The ontological and structural components of epistemic games. (Reproduced from Tuminaro and Redish [34])

Ontological Components		Structural Components	
Knowledge base	cognitive resources associated with the game	Entry and ending conditions	conditions for when to begin and end playing a particular game
Epistemic form	target structure that guides inquiry	Moves	activities that occur during the course of an e-game

per. Collins and Ferguson [27] outline several elements of epistemic games: entry conditions, moves, constraints, transfers, and a target epistemic form. Tuminaro and Redish [34] categorize components of e-games according to whether they are ontological or structural (see Table I). Unlike Collins and Ferguson, they give explicit attention to the "cognitive resources associated with the game," which they call the "knowledge base." Tuminaro and Redish also appear to group moves, constraints, and transfers into one component, which they call "moves."

The most important element of an epistemic game for Collins and Ferguson [27] is the epistemic form, which they define as "the target structures that guide scientific inquiry" and they identify different games primarily based on their target epistemic form. They use an analogy to tic-tac-toe to distinguish between the epistemic game and the epistemic form, where "the difference between forms and games is like the difference between the squares that are filled out in tic-tac-toe and the game itself."

Collins and Ferguson [27] introduce three broad categories of epistemic games — *structural analysis* games, *functional analysis* games, and *process analysis* games — and discuss several example games in each category. The canonical example that they use throughout the paper is the *list game*. This falls under the category of *structural analysis* games, where the primary goal is to determine the nature of some phenomenon "by breaking [it] down into subsets or constituents and describing the relationships among the constituents." Most of the games in this category focus on characterization and/or categorization.

In contrast, each of the epistemic games we present here belongs to the category of *functional analysis* games,[36] where "the goal is to determine the causal or functional structures that relate elements in a system."[27] In the context of the current study, this goal primarily manifests as an attempt to answer the following question: Which quantities depend on which other quantities; *i.e.,* which quantities are constant, which are independent variables, which are dependent variables, *etc.*?

In most disciplines, this functional structure is known and taking a partial derivative is simply a matter of playing Collins and Ferguson's *controlling-variables* game,

> "in which one tries to manipulate one variable at a time while holding other variables constant in order to determine the effect of each independent variable on the dependent variable."

However, one of the challenges of thermodynamics is that these relationships between quantities are not always readily apparent and thus, dealing with partial derivatives involves first playing a *functional analysis* game.

All of the games we discuss here attempt to answer the same question about the functional structure of a problem, but in fundamentally different ways. Although they share some moves and there is a large overlap in the requisite knowledge base, they differ significantly in their target epistemic form and in a few key moves. Our discussion primarily focuses on these differences since they raise some interesting questions that are relevant to contemplating thermodynamics instruction.

III. STUDY DESIGN

As mentioned earlier, this work arose from on-going discussions among three of the authors (CAM, TD, DR) regarding the possibility of using differentials as a bridge for understanding partial derivatives within thermodynamics. The first author (MBK) had not been a part of the earlier conversations and curriculum development. She conducted interviews with each of her three coauthors (who will be referred to by their initials) and seven additional faculty members from various institutions who have experience teaching thermodynamics (who will be referred to by pseudonyms). Table II describes the background of each of these experts. Each participant (except Tom[37]) was asked to think aloud as they worked through the following problem:

> Find $\left(\dfrac{\partial U}{\partial p}\right)_S$ for a van der Waals gas, given the following equations of state:

$$p = \frac{NkT}{V - N\,b} - \frac{a\,N^2}{V^2} \tag{1}$$

$$S = Nk\left\{\ln\left[\frac{(V - N\,b)T^{3/2}}{N\Phi}\right] + \frac{5}{2}\right\} \tag{2}$$

$$U = \frac{3}{2}NkT - \frac{a\,N^2}{V}. \tag{3}$$

Before proceeding to the rest of the paper, we suggest that the reader consider how s/he might approach this problem.

TABLE II. Background of interviewees. Although we only mention thermodynamics below, many of the courses taught by these participants include a significant statistical mechanics portion as well.(todo: how much of this info to include?)

Pseudonym	Institution Type	Discipline	Notes on relevant research or teaching experience
Leo	undergraduate	physics	Teaches upper-level undergraduate thermodynamics.
Chris	(non-traditional) undergraduate	physics	Teaches upper-level undergraduate thermodynamics in an interdisciplinary and non-traditional sequence.
Tom	undergraduate	physics	Physics Education Research on thermodynamics.
Matt	MS granting	physics	Physics Education Research on thermodynamics; Teaches upper-level undergraduate thermodynamics.
Jay	PhD granting	physics	Teaches upper-level undergraduate and graduate-level thermodynamics. (todo: relevant research?)
Gary	PhD granting	physics	Teaches upper-level undergraduate and graduate-level thermodynamics. (todo: relevant research?)
Keith	PhD granting	engineering	Engineering Education Research; Teaches Chemical Engineering Thermodynamics; Author of Engineering and Chemical Thermodynamics text.
DR	PhD granting	physics	co-author, Energy and Entropy instructor (todo: relevant research?)
CAM	PhD granting	physics	co-author, director of Paradigms in Physics Project
TD	PhD granting	math	co-author, director of Vector Calculus Bridge Project

Given the focus of the paper on understanding expert approaches to mathematical problems within thermodynamics, this problem was designed to have clear initial and final states (*i.e.,* the given values and the quantity to be found), but allow for multiple approaches. The goal was to have a problem similar to those students might encounter in an upper-division thermodynamics course, but which would be mathematically complex enough to be a challenge for the experts. Not counting co-authors, who had designed the problem, only Leo and Tom fully solved the problem with no errors, indicating that this task was indeed a problem for these experts.

In order to perform the cognitive task analysis, the first author (MBK) drew on video, transcripts, and written work from each interview, as well as both formal and informal follow-up conversations with the three co-authors. Descriptions of each game emerged by analyzing each experts' moves and stated justifications for those moves. For example, if an individual expert distinguished between two approaches, this was an indication that the individual viewed these as distinct activities. Similarly, if two experts used similar language to talk about similar moves, this was taken as an indication that they were engaged in the same game. Follow-up conversations with co-authors were used as additional check on the validity of these games.

We identified three primary epistemic games and two variations that these experts played or proposed while solving this problem. There were a few approaches that are not covered by these games (*e.g.,* work, heat capacities, *etc.*). However, we have chosen to report here on games that were played to completion and/or games that were proposed by a majority of the experts in this study.

IV. MATHEMATICS EPISTEMIC GAMES

As previously mentioned, the van der Waals problem presented above was designed to be a mathematical problem in a thermodynamics context and most of the experts treated it as such. The amount and type of physical reasoning the experts did varied from individual to individual. This sense-making activity will be partially discussed in Section VI, but will be more fully addressed in a future paper.

This section describes the three epistemic games — *Substitution*, *Partial Derivatives*, and *Differentials* — that these experts played or proposed while solving the van der Waals problem. The epistemic or target forms for these games and some of the key moves are summarized in Table III and will be discussed in detail below.

A. Substitution

The primary goal of the *Substitution* game is to use a set of functional relationships between certain quantities to discover the relationship between a different set of quantities. Typical moves within this game are to solve an expression for one variable in terms of another variable and to substitute one expression into another. Additionally, within the *Substitution* game, different constraints allow for different sets of moves. For example, if the given relationships are a set of **linear** equations, one can employ moves from linear algebra that are not appropriate for a set of **non-linear** equations.

One instantiation of the goal of the *Substitution* game for the van der Waals problem would be to take the given equations of state (Eqs. 1–3) that describe internal energy, pressure, and entropy as functions of volume and

TABLE III. Summary of target forms and key moves for each epistemic game

Game	Epistemic Form	Key Moves and Constraints
Substitution	$U = U(p, S)$	Isolate one variable as function of desired variables, Substitute one expression into another.
Partial Derivatives	$\left(\frac{\partial U}{\partial p}\right)_S = \left(\frac{\partial \square}{\partial \square}\right)_\square \left(\frac{\partial \square}{\partial \square}\right)_\square + ...$	Recursive use of partial derivative rules (Table IV) and/or Maxwell relations (Table VIII), Nice sets
Differentials	$dU = \left(\frac{\partial U}{\partial p}\right)_S dp + \left(\frac{\partial U}{\partial S}\right)_p dS$	Finding total differentials, linear algebra

TABLE IV. Several common partial derivative rules

Inversion:	$\left(\frac{\partial A}{\partial B}\right)_C = \left(\frac{\partial B}{\partial A}\right)_C^{-1}$
Cyclic chain rule:	$\left(\frac{\partial A}{\partial B}\right)_C = -\left(\frac{\partial A}{\partial C}\right)_B \left(\frac{\partial C}{\partial B}\right)_A$
1-dimensional chain rule [for $A = A(C, D)$ and $D = D(B, C)$] :	$\left(\frac{\partial A}{\partial B}\right)_C = \left(\frac{\partial A}{\partial D}\right)_C \left(\frac{\partial D}{\partial B}\right)_C$
2-dimensional chain rule [for $A = A(D, E)$ and both $D = D(B, C)$ and $E = D(B, C)$]:	$\left(\frac{\partial A}{\partial B}\right)_C = \left(\frac{\partial A}{\partial D}\right)_E \left(\frac{\partial D}{\partial B}\right)_C + \left(\frac{\partial A}{\partial E}\right)_D \left(\frac{\partial E}{\partial B}\right)_C$
2-dimensional chain rule (variant) [for $A = A(B, D)$ and $D = D(B, C)$]:	$\left(\frac{\partial A}{\partial B}\right)_C = \left(\frac{\partial A}{\partial B}\right)_D + \left(\frac{\partial A}{\partial D}\right)_B \left(\frac{\partial D}{\partial B}\right)_C$

temperature and find an explicit expression for internal energy as function of pressure and entropy. This expression for the internal energy is the target epistemic form for this game and is represented in Table III as $U = U(p, S)$. One possible set of moves to achieve this goal within this game is:

1. Solve Eq. 2, where $S = S(V, T)$, for $V(T, S)$.
2. Substitute $V(T, S)$ into Eq. 1, where $p = p(T, V)$, to get $p(T, S)$
3. Substitute $V(T, S)$ into Eq. 3, where $U = U(T, V)$, to get $U(T, S)$.
4. Solve $p(T, S)$ for $T(p, S)$
5. Substitute $T(p, S)$ into $U(T, S)$, to get $U(p, S)$.

Note that this game alone does not complete the task. However, once the internal energy is explicitly expressed as a function of pressure and entropy, one can take the desired partial derivative directly.

The van der Waals question had been deliberately chosen to make the *Substitution* game unattractively difficult. Indeed, this game was never on its own actually played to completion by any of these experts, usually because of a *clearly expressed* distaste for the algebra involved. For example, Matt described this game and his decision not to play it when he stated,

> *"So you've given me an analytical expression for S, so in principle I could plug that in, you know solve for something and plug that in there [U equation] and then I have an explicit S dependence, I would hold that [S] constant and take that derivative [$(\partial U/\partial p)_S$], but that seems like a real pain. So, I'm gonna put that aside for the moment, just because, I don't wanna do that right now."*

Although none of these experts played this game to completion, we include it here because all used the *Substitution* game as a sub-game within another game — this is what Collins and Ferguson [27] refer to as a "transfer": a move that involves transferring to a different game or sub-game. For all of the experts in this study, the goals and moves of the *Substitution* game were used to achieve a sub-goal within another game. It is important to note that in some preliminary student data, this is the only game that some of them consider.

B. Partial Derivatives

For the van der Waals problem, the *Partial Derivatives* game was the most common game played among this group of experts. Of the non-coauthor experts, all but Gary played this game to some extent.

The target epistemic form for this game is an expression that relates the desired partial derivative, $(\partial U/\partial p)_S$, to partial derivatives that can be directly calculated from given information, *i.e.*

$$\begin{aligned} &\left(\frac{\partial p}{\partial V}\right)_T, \left(\frac{\partial p}{\partial T}\right)_V, \\ &\left(\frac{\partial S}{\partial V}\right)_T, \left(\frac{\partial S}{\partial T}\right)_V, \\ &\left(\frac{\partial U}{\partial V}\right)_T, \left(\frac{\partial U}{\partial T}\right)_V. \end{aligned} \tag{6}$$

This target form is expressed in Table III as

$$\left(\frac{\partial U}{\partial p}\right)_S = \left(\frac{\partial \square}{\partial \square}\right)_\square \left(\frac{\partial \square}{\partial \square}\right)_\square + ...$$

where the right side of this equation is designed to represent the combination of derivatives from Eq. 6 for which one is looking. Keith summarized this game by stating,

> *"So I wanna use, be able to use these [derivatives in Eq. 6] to figure out mathematically how to relate U and p at constant S."*

Thus, unlike the *Substitution* game, the *Partial Derivatives* game bypasses the need to find an expression for U directly.

The key moves of this game are the various partial derivative rules (Table IV). There is one primary constraint on these moves: whether a move, or combination of moves, yields one or more of the derivatives from Eq. 6, which Jay refers to as "nice sets." Every expert who played this game evaluated each step to determine which derivatives could be calculated directly from the equations of state and which still needed to be rewritten in terms of those derivatives. In a previous paper,[38] we provided a detailed description of Jay's path through this problem while playing this game. A more idealized example of how one might play this game to solve the given problem is provided in Table V.

In order to completely answer the question, there is one more necessary move: to evaluate the derivatives in Eq. 6 for the given equations of state (Eqs. 1-3). This move can be played at any point in the game, but some experts did not use this move at all, stating as Jay did

> *"At this point, I have proven, reduced it to all kind of derivatives which I can take from there [points to equations of state], cause they have the right set, have the right combination of variables in there."*

Thus, at that point, he believed, as Keith did, that

> *"I have this thing solved in principle."*

Others (*e.g.,* Chris and Tom) only used this move after the target form had been obtained. Of those that played this game, only Leo made this move early. His solution will be discussed in more detail in Section V A.

C. Differentials

The *Differentials* game was far less common than the *Partial Derivatives* game — only the three co-authors played this game to completion without prompting. Some experts started this game and then abandoned it (*e.g.,* Keith, Matt, and Gary) and others played it only after being prompted (*e.g.,* Jay and Tom).

The target epistemic form for this game is:

$$dU = \left(\frac{\partial U}{\partial p}\right)_S dp + \left(\frac{\partial U}{\partial S}\right)_p dS, \tag{7}$$

where the solution to the van der Waals problem is simply the coefficient of the dp term.

The primary move for this game is fundamentally different than the moves in the *Partial Derivatives* game. One starts by finding the total differential for each equation of state:

$$dp = \left(\frac{\partial p}{\partial T}\right)_V dT + \left(\frac{\partial p}{\partial V}\right)_T dV \tag{8}$$

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV \tag{9}$$

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV. \tag{10}$$

Of those that played the *Differentials* game, some explicitly evaluated the derivatives,

$$dp = \left[\frac{2aN^2}{V^3} - \frac{NkT}{(V-Nb)^2}\right] dV + \frac{Nk}{V-Nb}\, dT \tag{11}$$

$$dS = \frac{Nk}{V-Nb}\, dV + \frac{3}{2}\frac{Nk}{T}\, dT \tag{12}$$

$$dU = \frac{aN^2}{V^2}\, dV + \frac{3}{2}Nk\, dT, \tag{13}$$

while others simply used placeholders for those derivatives (see Fig. 1).

One can then enter the *Substitution* game to solve the system of equations to get the target form, where the relevant quantities are now differentials instead of the variables themselves. Since this set of equations is **linear** in the differentials, one can either play the main *Substitution* game or the linear sub-game An idealized example, based on CAM's solution, is provided in Table VI.

The *Differentials* game, like the *Substitution* game, conceptualizes the goal in terms of solving a system of

TABLE V. Idealized example of the *Partial Derivatives* game.(todo: discuss)

1. Use 2D chain rule on the energy equation of state (Eq. 3), where $U = U(V, T)$.

$$\left(\frac{\partial U}{\partial p}\right)_S = \left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial V}{\partial p}\right)_S + \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial p}\right)_S \tag{4}$$

2. Rewrite the $(\partial T/\partial p)_S$ term

   (a) Use 2D chain rule variant on the $(\partial T/\partial p)_S$ term.

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial T}{\partial p}\right)_V + \left(\frac{\partial T}{\partial V}\right)_p \left(\frac{\partial V}{\partial p}\right)_S$$

   (b) Use the cyclic chain rule on the $(\partial T/\partial V)_p$ term and invert relevant terms.

$$\left(\frac{\partial T}{\partial V}\right)_p = -\left(\frac{\partial T}{\partial p}\right)_V \left(\frac{\partial p}{\partial V}\right)_T \Longrightarrow \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial p}{\partial T}\right)_V^{-1} - \left(\frac{\partial V}{\partial p}\right)_S \left(\frac{\partial p}{\partial T}\right)_V^{-1} \left(\frac{\partial p}{\partial V}\right)_T$$

3. Insert new expression for $(\partial T/\partial p)_S$ into Eq. 4, factor out the $(\partial V/\partial p)_S$ term and invert.

$$\begin{aligned}\left(\frac{\partial U}{\partial p}\right)_S &= \left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial V}{\partial p}\right)_S + \left(\frac{\partial U}{\partial T}\right)_V \left[\left(\frac{\partial p}{\partial T}\right)_V^{-1} - \left(\frac{\partial V}{\partial p}\right)_S \left(\frac{\partial p}{\partial T}\right)_V^{-1} \left(\frac{\partial p}{\partial V}\right)_T\right] \\ &= \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial p}{\partial T}\right)_V^{-1} + \left(\frac{\partial p}{\partial V}\right)_S^{-1} \left[\left(\frac{\partial U}{\partial V}\right)_T - \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial p}{\partial T}\right)_V^{-1} \left(\frac{\partial p}{\partial V}\right)_T\right]\end{aligned}$$

   At this point, everything except $(\partial p/\partial V)_S$ is a derivative that can be calculated directly from the equations of state.

4. Rewrite the $(\partial p/\partial V)_S$.

   (a) Use 2D chain rule variant on the $(\partial p/\partial V)_S$ term.

$$\left(\frac{\partial p}{\partial V}\right)_S = \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S$$

   (b) Use the cyclic chain rule on the $(\partial T/\partial V)_S$ term and invert relevant terms.

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial T}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T \Longrightarrow \left(\frac{\partial p}{\partial V}\right)_S = \left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial S}{\partial T}\right)_V^{-1} \left(\frac{\partial S}{\partial V}\right)_T$$

5. Insert new expression for $(\partial p/\partial V)_S$ into the expression found in Step 3 and simplify.

$$\left(\frac{\partial U}{\partial p}\right)_S = \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial p}{\partial T}\right)_V^{-1} + \left[\left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial S}{\partial T}\right)_V^{-1} \left(\frac{\partial S}{\partial V}\right)_T\right]^{-1} \left[\left(\frac{\partial U}{\partial V}\right)_T - \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial p}{\partial T}\right)_V^{-1} \left(\frac{\partial p}{\partial V}\right)_T\right]$$

$$\left(\frac{\partial U}{\partial p}\right)_S = \frac{\left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial S}{\partial T}\right)_V - \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial S}{\partial V}\right)_T}{\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial S}{\partial T}\right)_V - \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial S}{\partial V}\right)_T} \tag{5}$$

equations. However, using total differentials linearizes the system of equations, providing the option of the linear sub-game for solving the system. Additionally, like the *Partial Derivatives* game, using total differentials bypasses the need to find an expression for U explicitly.

On the other hand, the primary constraint on moves in the *Partial Derivatives* game — whether a move yields derivatives that can be directly evaluated from the equations of state — is automatically satisfied in the Differentials game. By taking the equations of state and "zapping

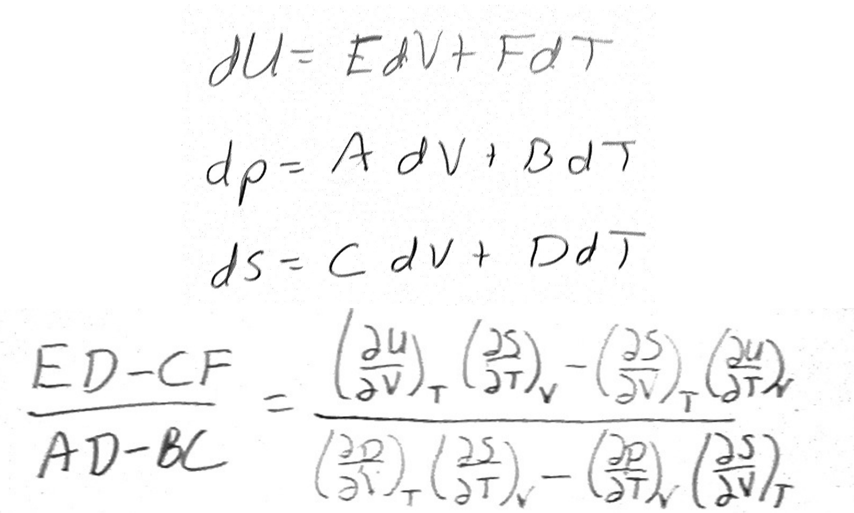

FIG. 1. CAM's use of letters as placeholders for explicit derivatives and her solution (see Table VI).

with d," one is working only with the derivatives in Eq. 6 ("nice sets"), which eliminates the need to evaluate each step for this constraint.

V. GAME VARIATIONS

There are two variations that can be played within the games discussed above. These variations can be thought of as possible moves that provide alternate paths through each game. Typically, these alternate path manifest by providing additional constraints on remaining moves and/or in producing a variant to the epistemic form. This section will briefly describe each variation and how it impacts the different games.

A. Constant Entropy as Constraint

The first variation involves explicitly setting entropy equal to a constant and using this constraint to reduce the degrees of freedom for the problem.

In the *Differentials* game, this move involves setting $dS = 0$ explicitly, which both TD and Jay did in the total differential form of the entropy equation of state (Eq. 2),

$$dS = 0 = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV. \qquad (16)$$

This move produces a variation in the epistemic form by eliminating one of the degrees of freedom. Thus, instead of looking for an expression for dU in terms of dp and dS, one is looking for an expression for dU in terms of the *single* variable, dp that is valid only for constant entropy. One can also explicitly hold the entropy constant by setting $dS = 0$ in the thermodynamic identity, but since this approach combines two different variations, it will be discussed more fully in Section V B.

In the *Substitution* game, this variation can serve as one of several possible starting moves. By explicitly setting the entropy equation of state,

$$S = Nk\left\{\ln\left[\frac{(V - N\,b)T^{3/2}}{N\Phi}\right] + \frac{5}{2}\right\} \qquad (2)$$

equal to a constant, one can find an explicit relationship between T and V. This relationship can, in turn, be used to find U and p as functions of only one variable, *e.g.,* $U = U(V)$ and $p = p(V)$. As in the *Differentials* game, solving this new set of equations leads to a variant of the original epistemic form, an expression for internal energy as a function only of pressure, *i.e.,* $U = U(p)$, that is only valid for constant entropy.

In the *Partial Derivatives* game, one can use this variation by entering the *Substitution* game as described above to find internal energy and pressure as functions of only one variable. This then places constraints on the remaining moves. Leo, whose written work is shown in Figure 2, was the only expert to use this move within the *Partial Derivatives* game. He began by applying the 1-dimensional chain rule,

$$\left(\frac{\partial U}{\partial p}\right)_S = \left(\frac{\partial U}{\partial V}\right)_S \left(\frac{\partial V}{\partial p}\right)_S. \qquad (17)$$

However, he added a "+" to this equation, stating "*so the other one would be the...T term.*" He seemed to be conflating the 1-dimensional chain rule and the 2-dimensional chain rule (see Table IV).

Instead of pursuing the T term, he focused instead on the issue of constant entropy, asking himself aloud, *"How do [I] keep entropy constant?"* to which his answer was to explicitly set Eq. 2 equal to a constant:

TABLE VI. Idealized example of the *Differentials* game (based on CAM's solution).

1. Find the total differential for each equation of state (Eqs. 1–3), either explicitly evaluating the derivatives or using placeholder names, as below.

$$dp = \left(\frac{\partial p}{\partial V}\right)_T dV + \left(\frac{\partial p}{\partial T}\right)_V dT = A\,dV + B\,dT \tag{8}$$

$$dS = \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial S}{\partial T}\right)_V dT = C\,dV + D\,dT \tag{9}$$

$$dU = \left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT = E\,dV + F\,dT \tag{10}$$

2. Multiply Eq. 8 by D and Eq. 9 by $-B$, then add to find dV in terms of dp and dS.

$$D\,dp = +AD\,dV + BD\,dT$$
$$-B\,dS = -BC\,dV - BD\,dT$$

$$\Longrightarrow D\,dp - B\,dS = (AD - BC)\ dV$$

$$dV = \frac{D\,dp - B\,dS}{AD - BC} \tag{14}$$

3. Multiply Eq. 8 by $-C$ and Eq. 9 by A, then add to find dT in terms of dp and dS.

$$-C\,dp = -AC\,dV - BC\,dT$$
$$A\,dS = +AC\,dV + AD\,dT$$

$$\Longrightarrow -C\,dp + A\,dS = (AD - BC)\ dT$$

$$dT = \frac{-C\,dp + A\,dS}{AD - BC} \tag{15}$$

4. Substitute Eqs. 14 and 15 into Eq. 10 to get an expression for dU in terms of dp and dS.

$$dU = E\left(\frac{D\,dp - B\,dS}{AD - BC}\right) + F\left(\frac{-C\,dp + A\,dS}{AD - BC}\right) = \left(\frac{ED - FC}{AD - BC}\right) dp + \left(\frac{FA - EB}{AD - BC}\right) dS$$

5. At this point, the solution is simply the coefficient of the dp term (see Fig. 1), which is equivalent to the solution from the *Partial Derivatives* game (see Eq. 5).

$$\left(\frac{\partial U}{\partial p}\right)_S = \frac{ED - FC}{AD - BC} = \frac{\left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial S}{\partial T}\right)_V - \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial S}{\partial V}\right)_T}{\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial S}{\partial T}\right)_V - \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial S}{\partial V}\right)_T}$$

"Ok, so this [Eq. 2] *tells me, if I want to hold the entropy constant, then S,*

$$S\,(\text{constant}) \longrightarrow$$

S constant implies, [points at various parts of Eq. 2] *constants, constants, constants. Ok, so, this guy* [points to numerator of logarithm in Eq. 2] *is constant.*

$$(V - Nb)T^{3/2}\,(\text{constant}) \tag{18}$$

Ok, so, if this is constant, then I can say, so this equals, capital C isn't taken, so

$$= C$$

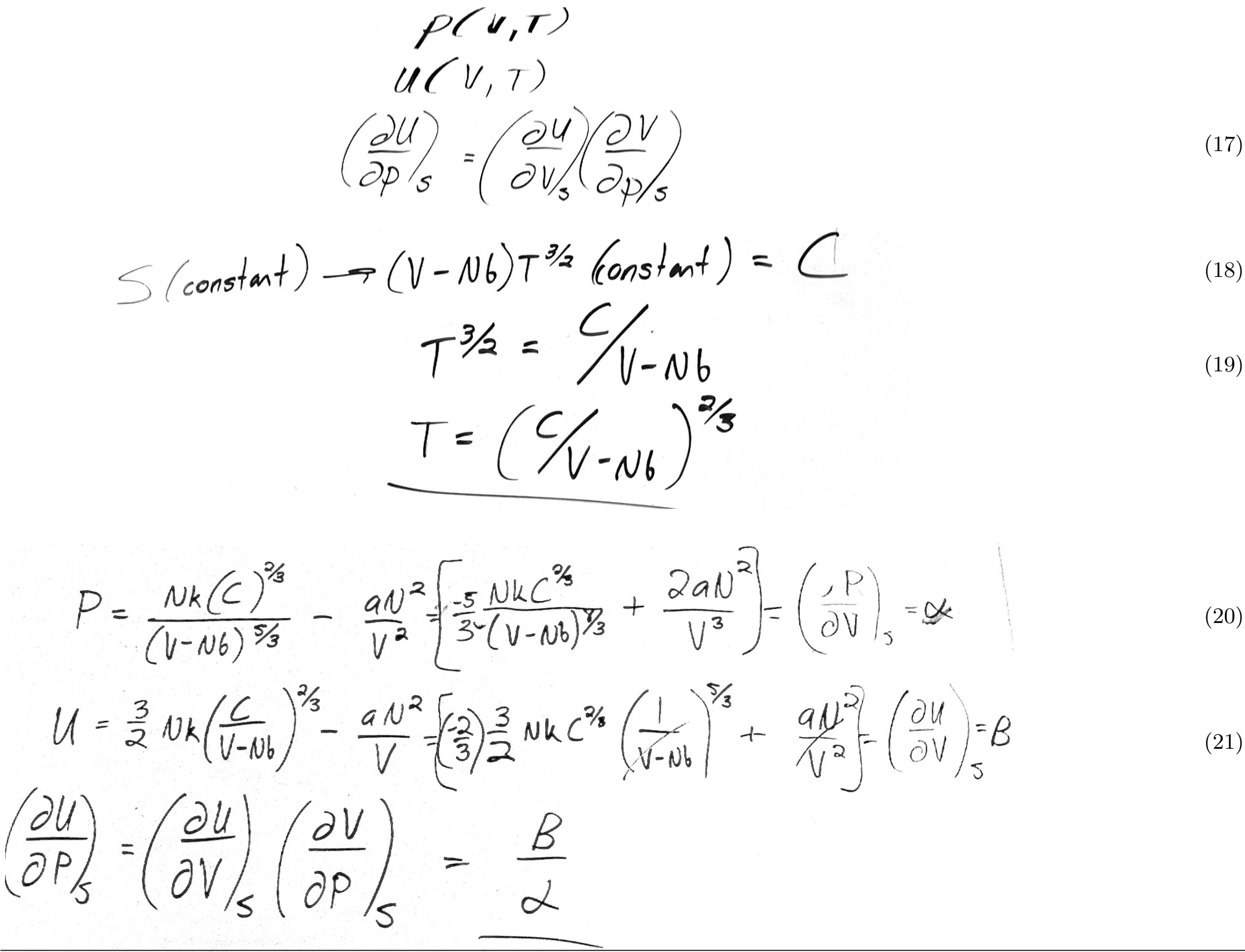

FIG. 2. Leo's written work that demonstrates the use of constant entropy as a constraint (Section V A).

> *So this tells me that*
>
> $$T^{3/2} = \frac{C}{V - Nb}. \tag{19}$$
>
> *As long as this is true, then I feel like I'm confident that things are constant entropy..."*

Leo then went on to outline his plan to solve for U and p as functions of only one variable, V:

> *"...So what I want to do is basically pick to solve this* [points to Eq. 19] *for either T or V and then I want to put that in here* [points to U equation of state, Eq. 3] *so that U and p* [points to p equation of state, Eq. 1] *both become a constant [sic]. So I can basically get rid of the T dependence and calculate. So, now p is function of V, and what that is, is p as a function of V for a constant entropy process. And then I can do the same with U, U is a function of, let's say V, for a constant entropy process..."*

After carrying out this plan and calculating both $p(V)$ and $U(V)$,

$$p = \frac{Nk(C)^{2/3}}{(V - Nb)^{5/3}} - \frac{aN^2}{V^2} \tag{20}$$

$$U = \frac{3}{2}Nk\left(\frac{C}{V - Nb}\right)^{2/3} - \frac{aN^2}{V}, \tag{21}$$

Leo evaluated the derivative of each with respect to V and returned to the chain rule (Eq. 17) to calculate a final answer to the problem (see Fig. 2).

Explicitly setting the entropy equal to a constant ultimately resolved Leo's conflation of the 1-dimensional and 2-dimensional chain rules. However, this conflation appeared to cause him to question the "legality" of his math throughout his interview (this will be discussed further in Section VI B).

In all cases, the primary indicator of Variation 1 is a move that explicitly sets the entropy equal to a constant (or the change in entropy equal to zero). This move,

TABLE VII. Thermodynamic Potentials

Name	Fundamental Equation
Internal Energy	$dU = T\,dS - p\,dV$
Enthalpy	$dH = T\,dS + V\,dp$
Helmholtz Free Energy	$dF = -S\,dT - p\,dV$
Gibbs Free Energy	$dG = -S\,dT + V\,dp$

and consequently this variation, reduces the degrees of freedom and produces a variant in the target epistemic form, regardless of which game one is playing.

B. Thermodynamic Potentials

The second variation involves using a total differential of a thermodynamic potential, either the thermodynamic identity,

$$dU = T\,dS - p\,dV, \tag{22}$$

or one of its Legendre transforms (see Table VII).

In the *Differentials* game, using a thermodynamic potential changes the set of linear equations one is trying to solve. For example, DR used the thermodynamic identity (Eq. 22) instead of the differential form of the equation of state (Eq. 10). (todo: talk to CAM re: Tom's solution) Tom used this variation when he equated the thermodynamic identity with the target form for the *Differentials* game (Eq. 7):

$$dU = T\,dS - p\,dV = \left(\frac{\partial U}{\partial p}\right)_S dp + \left(\frac{\partial U}{\partial S}\right)_p dS. \tag{23}$$

Substituting in the differential forms for the pressure and entropy equations of state (Eqs. 8 and 9), he separately equated the resulting dT and dV components:[39]

$$\left(\frac{\partial U}{\partial p}\right)_S \left(\frac{\partial p}{\partial T}\right)_V + \left(\frac{\partial U}{\partial S}\right)_p \left(\frac{\partial S}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \tag{24}$$

$$\left(\frac{\partial U}{\partial p}\right)_S \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial U}{\partial S}\right)_p \left(\frac{\partial S}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p. \tag{25}$$

He then solved this set of equations for the desired derivative, $(\partial U/\partial p)_S$.

In the *Partial Derivatives* game, using the thermodynamic identity changes the goal of the problem. From the thermodynamic identity, one can see that

$$\left(\frac{\partial U}{\partial V}\right)_S = -p. \tag{26}$$

When this is used with the 1-dimensional chain rule,

$$\left(\frac{\partial U}{\partial p}\right)_S = \left(\frac{\partial U}{\partial V}\right)_S \left(\frac{\partial V}{\partial p}\right)_S, \tag{27}$$

one essentially replaces the original goal, finding $(\partial U/\partial p)_S$, with the goal of finding $(\partial V/\partial p)_S$:

$$\left(\frac{\partial U}{\partial p}\right)_S = -p\left(\frac{\partial V}{\partial p}\right)_S. \tag{28}$$

One can also get to Eq. 28 by dividing the thermodynamic identity (Eq. 22) by dp and using Variation 1 to eliminate the dS term. Matt and Jay did this explicitly, but Keith did so more implicitly. While considering the thermodynamic identity, he said

> So, if I look at $\partial U/\partial p$ at S, I would have again $\partial S/\partial p$ at S, is zero and I would have, minus p times partial of V with respect to p at S [writes Eq. 28].

Jay explicitly identified $(\partial V/\partial p)_S$ as the adiabatic compressibility and all moves after this one referred to finding the adiabatic compressibility as the goal.[38]

Matt described this variation as a different way of constructing one's calculus:

> *"If I think about this, the first law implies, says that U is my dependent variable, and I sort of think of S and V as being my axes... Those are independent variables and so then I can sort of construct my calculus in that way."*

In addition to potentially changing the goal, using a thermodynamic potential within the *Partial Derivatives* game opens up additional moves through the use of Maxwell relations (Table VIII). Several experts (*e.g.,* Matt, Tom, DR) mentioned or proposed using Maxwell relations, although Keith was the only one to actually use them in his final solution.

Gary spent the majority of his time considering the various thermodynamic potentials (Table VII) and

TABLE VIII. Maxwell relations.

$$\frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial \Phi}{\partial x_j}\right)$$

Four common Maxwell relations:

$$\frac{\partial^2 U}{\partial S\,\partial V} = +\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$

$$\frac{\partial^2 H}{\partial S\,\partial p} = +\left(\frac{\partial T}{\partial p}\right)_S = +\left(\frac{\partial V}{\partial S}\right)_p$$

$$\frac{\partial^2 A}{\partial T\,\partial V} = +\left(\frac{\partial S}{\partial V}\right)_T = +\left(\frac{\partial p}{\partial T}\right)_V$$

$$\frac{\partial^2 G}{\partial T\,\partial p} = -\left(\frac{\partial S}{\partial p}\right)_T = +\left(\frac{\partial V}{\partial T}\right)_p$$

weighing the advantages and disadvantages of using each. He was one of the few experts that did not actually get to a solution.

Employing a thermodynamic potential involves bringing in physical information in addition to the mathematical relationships between total differentials and partial derivatives or the partial derivative rules. In this sense, using this move implied a broader knowledge base than was strictly necessary to solve the problem mathematically. It also more firmly grounds the problem in physical context than any of the previously discussed moves by connecting particular partial derivatives to specific thermodynamic variables.

VI. DISCUSSION

It is clear that there are several points of commonality amongst the epistemic games and variations discussed in this paper. As mentioned in Section II B, each of these games involves an attempt to answer the question:

> Which quantities depend on which other quantities (*i.e.,* which quantities are constant, which are independent variables, which are dependent variables, *etc.*)?

and as such, can be considered what Collins and Ferguson [27] call *functional analysis* games.

In a similar way, each of these games seem to share an entry condition: to notice a mismatch between the functional relationships provided and the one required to solve the problem, *i.e.,* the given equations of state are in terms of one set of variables (V and T) and the problem asks for a partial derivative with respect to different variables (p and S). Although the recognition of this mismatch appears to trigger each of these games, it is unclear from these interviews why experts choose to initially engage in one game or variation over another. We suspect that there may be additional entry conditions that are more associated with previous experience and beliefs about thermodynamics. However, one would have to delve more deeply to uncover these connections.

There is also significant overlap in the requisite knowledge base, *i.e.,* the cognitive and mathematical resources that are necessary to play the game. For example, the *Substitution* game is necessary as a sub-game in both of the other games. Also, although it may be implemented at different points in each game, one must be able to actually evaluate a partial derivative. Perhaps most importantly, one must be able to recognize the multivariable nature of the problem.

Despite these commonalities, the games diverge in interesting ways. In this section, we explore more fully some of the implications of these differences, particularly between the *Partial Derivatives* game the *Differentials* game. We discuss how the difference in epistemic form implies a different conceptualization of partial derivatives (Section VI A). We also highlight expert concerns about the "legality" of the mathematics in the two different games and relate this to differences between mathematicians and physicists in their use of partial derivatives and differentials (Section VI B).

In addition to contrasting these games to each other, we present some reflections from these experts on the nature of thermodynamics and how they connect to some of the criteria they used to evaluate their progress and solutions (Section VI C). Finally, we consider how the games presented here compare to other previously identified epistemic games (Section VI D).

A. Concept of partial derivatives

As shown in Table III, one of the most notable differences between these games is in the target epistemic form. For example, the goal of the *Substitution* game (when played by itself) is to find and differentiate an explicit expression for internal energy as a function of pressure and entropy. Yet both the *Partial Derivatives* game and the *Differentials* game bypass this goal through differentiation.

The difference in epistemic form between the *Partial Derivatives* game and the *Differentials* game suggests a more fundamental difference in how one conceptualizes a partial derivative.

After being prompted to use total differentials, Jay reflected that

> *"...essentially this is a different encoding of the same information. So, if you think about it, I mean, I don't see off-hand any reason why this encoding is different from writing as partial derivatives. I mean the form is different, but I think the encoding is the same."*

At the prompting of the interviewer, Jay went on to more clearly distinguish between these two forms.

> *"If you think about small changes*

$$\Delta x, \Delta y, \Delta z \tag{29}$$

> *whatever they are, you find equations that relate those things... you write... the changes as changes in themselves and not how they're related to each other cause that you solve [for]... later on by picking out which of these [changes] are constant. I mean the other approach, where you write things like*

$$\left(\frac{\partial x}{\partial y}\right)_z \tag{30}$$

> *...here [points to Eq. 29] you work with variables, here [points to Eq. 30] you work with ratios of variables, directly, but you have to make sure you pick the right ratios.*

So, you have the same information. Here [points to Eq. 29] you have the independent changes and... you have the freedom to connect them somehow. Here [points to Eq. 30] you have them as dependent changes because you say ok there's a, out of these [Eq. 29] I can construct... three different ratios... and I'm not sure which one is the best.

Jay is clearly distinguishing between what a partial derivative is in each game. In the *Differentials* game, a partial derivative is a ratio of small independent changes, whereas in the *Partial Derivatives* game, it is one particular dependent change. While both contain the same information, they are "encoded" differently.

This difference in how these two game conceptualize a partial derivative is particularly relevant to thinking about what we want our students to be able to do. The theoretical framework proposed by Zandieh[17] and applied to student understanding of ordinary derivatives would be an excellent way to explore this area further. (todo: discuss with CAM)

B. "Mathematically Illegal"

Several experts expressed the idea that one or more of their moves might not be considered kosher to a mathematician. For example, as pointed out in Section V A, Leo questioned whether the valid 1-dimensional chain rule he wrote,

$$\left(\frac{\partial U}{\partial p}\right)_S = \left(\frac{\partial U}{\partial V}\right)_S \left(\frac{\partial V}{\partial p}\right)_S, \tag{17}$$

was "mathematically legal" due to the lack of explicit temperature dependence:

"So, I feel like what I'm writing [Eq. 17] is not quite mathematically, sort of, legal, but I think it's, but I think it's physically legal...the legal illegal part is why I don't have a $\partial U/\partial T$ term in here and I think that, I feel like that's ok because I'm thinking about a process, so I'm really thinking about small changes in each of these [gestures to ∂U and ∂p] as entropy [points to S] stays constant. So, I think that's fine because these are both [points to $p(V)$ and $U(V)$, Eqs. 20 and 21], the temperature changes are built into these, both of these pieces."

He did not at first seem to realize that in the 1-dimensional chain rule he had written, the temperature dependence was buried in the constant entropy. Although he ultimately resolved the issue by explicitly holding the entropy constant, he continued to question the legality of his mathematics.

We believe that his conflation of the two chain rules may be related to an issue that was raised in Section II B regarding the unique role of thermodynamics when it comes to the *controlling-variables* game. Collins and Ferguson[27] define this game as one

> "in which one tries to manipulate one variable at a time while holding other variables constant in order to determine the effect of each independent variable on the dependent variable."

and this game is introduced as early as elementary school.[40] When learning about partial derivatives in a multivariable calculus class, this game is typically interpreted to mean holding "everything else" constant. However, one cannot always hold **everything else** constant. For example, in Galileo's classic rolling ball experiment, one can change the mass of the balls by changing the size or the density, but you cannot change the mass and hold both size and density constant. Thus, a more nuanced interpretation of this game is to hold constant **as many variables as possible**. In most disciplines, including mathematics, the number of variables is typically known, as well as which are dependent and independent variables. Thus, one can use the less nuanced interpretation. In thermodynamics, these functional relationships are not always readily apparent. Therefore, one must have a more nuanced perspective and *functional analysis* games like those described here are vital to understanding the situation.

(todo: Discuss and clarify) In the *Differentials* game, the issue of what to hold constant is not as problematic, since one first evaluates a total differential and only later decides what to hold constant (*e.g.,* by setting $dS = 0$). However, the *Differentials* game raises other issues related to the "legality" of mathematics. For example, Matt referred to dividing by a differential as "that bogus physicist-y way" of turning a differential into derivative:

"So now I have dU is minus $p\,dV$ and then I thought about trying to make that into a partial derivative in sort of that ***bogus physicist-y way*** *by treating it as a fraction and then just ignoring the fact that it's, I mean just sort of changing that fraction into a derivative in some sort of sneaky way. So on the left-hand side I have partial $\partial U/\partial p$ with S held constant and then I have to sort of take the other side, over dp basically. And I have to be a little careful here, I think because, I have to use, do I have to use some sort of product rule thing here? Huh. [long pause]."*

(todo: Discuss and clarify) Unlike mathematicians, physicists are willing to work with differentials as intuitive objects standing for small changes.[21] Yet, just as with the issue of what to hold constant, physicists are not always sure of the legitimacy of their use of differentials from a mathematics perspective.

Given the disconnect between mathematics and physics use as discussed above, physicists clearly do not learn these views in their math courses, which implies that they are passed on in a physics context instead. It should therefore not be surprising that the language and notation that physicists have developed to deal with partial derivatives in thermodynamics may seem to be at odds with a pure mathematics approach. Nor should we be surprised that students struggle to make sense of the complex functional structures within thermodynamics, given the nuance required. Two of the co-authors (TD and CAM) are currently working on project that is more fully exploring the impact of the disconnect between mathematicians and physicists on the use of differentials in the classroom.

C. Nature of thermodynamics

Several experts commented on the fact that it was possible to find a solution that was independent of the specific functional form of the equations of state. Those that did seemed to feel that this characteristic distinguished thermodynamics from other sub-disciplines in physics. According to Matt,

> *"...the place where we get Maxwell relations and all the partial derivative stuff that you do in thermo, to me,... it's just calculus, if you have multi-variable calculus... and it's not specific to any system."*

And Chris pointed out,

> *"One of the challenges... and it's not a challenge, it's the power of thermodynamics,... the formalism is independent of any underlying model... It's a different way of thinking about it and so, you know, in that sense, it's a little bit unappealing for lots of people... but... classical thermo is a beautiful theory."*

This view of thermodynamics as an abstract mathematical formalism is consistent with the ways that most of these experts treated this problem. For example, when evaluating their progress, they tended to focus on aspects related to the mathematical structure of the problem such as whether a given step produced a "nice set."

Despite statements like those above, when it comes to the nature of thermodynamics, these experts seem to simultaneously hold two perspectives in tension. This was evident in how they attempted to evaluate their solutions, as opposed to their progress. Instead of the mathematical structure, they focused on the physical model and physical intuition. For example, after trying one approach, Matt decided to evaluate his expectations:

> *"I'm gonna stop for a second and just think about what, I expect to happen. Maybe I should have done that earlier. So, if I did the experiment I did in one, insulated, slow, no heat transfer, add mass to it, I do work on the gas, um, as I increase the pressure, the volume's going to decrease. [pause] It's a van der Waals gas, so two competing effects, one of which is that the particle's solid. If I get to a sufficiently dense gas, I have to worry about the actual volume of the particles themselves. That is what little b means and then I also have to think about the fact that there's an attractive force between the particles. So, if the volume decreases, what'll I expect to happen to the internal energy? If there was no interactions between the particles, it would of course increase cause I do positive work on the gas. Well, heck, what am I thinking, so if I [do] positive work on the gas, the internal energy has to increase. That's not ambiguous, I should know that."*

Similarly, both Leo and Chris attempted to evaluate their solution in terms of a limiting case (*i.e.,* an ideal gas). However, Leo was unsatisfied with not having an intuition for what the answer should be in this case,

> *"Oh, I'm just, I'm having trouble convincing myself that this answer [for an ideal gas], I mean it seems reasonable, but it, but I don't have a good way of saying, and I mean I guess [what] I could do is look back to my process and say, do I believe all the steps? Um, I'm, you know, I'm slightly comforted by the fact that the units work out right, um, but I'd like some kind of other support."*

One of the few exceptions to this dichotomy was DR's use of dimensional analysis as a way to evaluate his progress. (todo: add more or take this out)

Each of these statements seem to suggest that there are two aspects to thermodynamics, which these experts move between fluidly. On the one hand, there is the abstract mathematical formalism that allows one to derive general relationships that will hold regardless of the physical system. On the other hand, evaluation of the meaning or the correctness of a solution is usually based more on what that formalism tells us about a specific system (such as an ideal or van der Waals gas). Understanding expert views about the nature of thermodynamics and how these views are held in tension has significant implications for our instruction and for helping our students to develop professionally.

D. Comparison to other Epistemic games

We have already discussed how the games presented in this paper compare to those identified by Collins and Ferguson.[27] The games identified by Lunk[29] are particular to computational modeling and so are not applicable to the task in this paper. The *Answer-Making Epistemic Game* that Chen *et al.*[28] describes does not appear to

TABLE IX. List of epistemic games identified in Tuminaro and Redish.[34]

Name of the Game	Description of the Game
Mapping Meaning to Mathematics	"In this game, students begin from a conceptual understanding of the physical situation described in the problem statement, and then progress to a quantitative solution."
Mapping Mathematics to Meaning	"In this game, students develop a conceptual story corresponding to a particular physics equation."
Physical Mechanism Game	"In the Physical Mechanism Game students attempt to construct a physically coherent and descriptive story based on their intuitive sense of physical mechanism."
Pictorial Analysis	"In the Pictorial Analysis Game, students generate an external spatial representation that specifies the relationship between influences in a problem statement."
Recursive Plug-and-Chug	"In the Recursive Plug-and-Chug Game students plug quantities into physics equations and churn out numeric answers, without conceptually understanding the physical implications of their calculations."
Transliteration to Mathematics	"Transliteration to Mathematics is an epistemic game in which students use worked examples to generate a solution without developing a conceptual understanding of the worked example."

be characteristic of these experts. In contrast, there are connections to three of the six games that Tuminaro and Redish[34] claim are characteristic of introductory physics students (see Table IX).

One could argue that the games in this paper are all simply instances of what Tuminaro and Redish[34] call the *Recursive Plug-and-Chug* game, but according to them,

> "...students playing this game rely only on their syntactic understanding of physics symbols, without attempting to understand these symbols conceptually."

It is true that some of these experts did not always connect mathematical quantities to the physical situation. However, they did connect these quantities to their conceptual understanding of the mathematics, which was fairly sophisticated and nuanced in some cases, and to the mathematical structure of the problem. In fact, the constraints that these experts place on their mathematical moves, particularly in the *Partial Derivatives* game, clearly distinguish their activities from the *Recursive Plug-and-Chug* game, which involves little to no evaluation of one's moves. In this sense, their actions are more akin to the *Mapping Meaning to Mathematics* game or the *Mapping Mathematics to Meaning* game, where the story they are telling is a mathematical story and not necessarily a physical one.

Both the *Mapping Meaning to Mathematics* game and the *Mapping Mathematics to Meaning* game represent high level games that involve the interaction between the physical situation and the mathematics used to understand it. Some of the experts in this study seemed to be playing one of these games, where the *Partial Derivatives* game or the *Differentials* game were played as a sub-game (see Section VI C for examples of experts making connections to the physical situation).

VII. CONCLUSIONS

In this paper, we have used epistemic games as an analytic tool for performing cognitive task analyses looking at expert performance. Employing this tool, we have performed a task analysis on a mathematical problem in thermodynamics and introduced three epistemic games and two variations that represent expert approaches to this problem. This approach has led to several interesting insights into expert use and understanding of the mathematics of thermodynamics and to several areas for further exploration.

In this section, we discuss some of the limitations of this study, briefly summarize some of the major implications of this work and some current areas of research that have grown from it, as well as suggest some further lines of study.

A. Limitations

As with all research, there are some limitations to the conclusions that we can draw from the data in this study.

Although we attempted to recruit faculty with expertise in thermodynamics, we cannot evaluate the representativeness of our sample.

Additionally, we recognize the inherent socio-cultural nature of practice and that for physics professionals, problem solving is often a collaborative exercise. This aspect of practice was not represented in these individual interviews, but the use of epistemic games as an analytic tool is equally as valuable (if not more so) in a setting which does focus on the socio-cultural aspects of problem solving.

We intentionally posed a problem that is inherently mathematical in nature and had clear initial and final conditions. Although more challenging for experts than introductory-level "end-of-chapter" problems, it is still artificial. This is also not the only kind of problem in thermodynamics and perhaps not even the most important. Since it was inherently mathematical and these experts treated it this way, the analysis conducted here has more to say about how physicists understand and use the mathematics of thermodynamics than about how they conceptualize the physics of thermodynamics: an important distinction that must be explored.

In categorizing the behavior of these experts through the use of epistemic games, we do not claim that those discussed here are a complete set. There were moves that some experts made that were not discussed here, primarily because the experts themselves chose to abandon these approaches fairly early in the process. Some of these moves may be part of the games discussed here or may constitute other games altogether. In order to explore them more fully, one must provide tasks that focus on different aspects of thermodynamics.

For example, some of the experts (Leo, Matt, Keith, and Chris) answered another prompt before the van der Waals question:

> Draw and describe an experiment that would measure the quantity:
>
> $$\left(\frac{\partial U}{\partial p}\right)_S .$$

This type of question is used extensively in the Energy and Entropy course[18] at Oregon State University and these "Name the Experiment" activities are described in Roundy *et al.*.[41] This prompt is designed to cue a process where the primary goal is to answer the question "How can the quantity X (in this case, the derivative $(\partial U/\partial p)_S$) be physically measured?" As Matt pointed out in answering this question,

> ***"This is an adiabatic process and so I can figure out, basically the change in internal energy is gonna be equal to the work done*** *and so I can take integral $p\,dV$, um, stick a negative sign in front of it and that would give me the work, which would also be the change in internal energy."*

Both Leo and Matt began the van der Waals problem by returning to this idea of finding the work done, both explicitly mentioning the previous problem. Although Matt pursued this path to the point of attempting to evaluate an integral, both he and Leo ultimately recognized that the van der Waals problem is not particularly soluble with this approach and abandoned it in favor of the *Partial Derivatives* game discussed in this paper.

The fact that the earlier problem cued an approach that was in this case unproductive, is important confirmation that performance is to some extent context-dependent. It would be worthwhile to consider what experts do with a problem where the kind of physical reasoning that is important in the Name the Experiment question would also be helpful in the kind of mathematical problem discussed in this paper.

B. Implications and Future work

In our discussion in Section VI, we presented three significant areas of interest that emerged from a comparison of the games presented in this paper. Here we briefly revisit these areas and discuss current and future work exploring this issues further.

The *Partial Derivatives* game and the *Differentials* game represent two very different ways of conceptualizing a partial derivative. The recognition of this difference has prompted a study that more fully explores how experts in different disciplines understand partial derivatives. We are now in the process of conducting group interviews where experts in different STEM fields explore the use of partial derivatives in their field. We intend to use this research to inform curriculum development in both mathematics and upper-division physics.

The disconnect between mathematics and physics was apparent in how some of these experts discussed the "legality" of their mathematics and raised an important issue regarding where and how our students learn to play these games. As Morrison and Collins[30] point out,

> "...you learn how to play...simply and only by playing these games with people who are already relatively more fluent than you are — and who, crucially, are willing to gradually pull you up to their level of expertise by letting you play with them."

The issue related to holding "everything else" constant has led to the development of an apparatus designed to be a mechanical analogue of a thermodynamic system that can allow students to explore the impact of which quantities are held constant.[42]

The experts in this study clearly recognize the complexity and beauty of the mathematical formalism of thermodynamics. Yet, they have also developed the ability to focus on the conceptual and physical story when needed. This interplay between complex mathematical structure and physical understanding and intuition is particularly important to cultivate in a field like thermodynamics where students often cannot make the connections between the unfamiliar mathematical techniques and the unfamiliar physical quantities like entropy. We plan to explore how activities designed to provide more physical significance to thermodynamics through concrete examples[41–44] can help students to make connections between the mathematics and the physics.

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