Exploring the *Gradient*

This narrative provides an example of instruction introducing upper-level physics students to the meaning of the mathematical entity known as *the gradient* in describing physical phenomena. This class session occurred during Physics 320, Symmetries and Idealizations, the first in a sequence of nine "paradigms physics" courses for juniors.[1]

Figure 1. Discussion of concept of *gradient* in describing physical phenomena

The interpretative narrative[2] below is based upon the video and transcript of a small group activity and whole class discussion about the students' use of innovative materials to explore the concept of the gradient. The innovative materials included clear plastic surfaces that represent the gradient of a function of two variables, related topographical maps, sheets with coordinate grids, and a tool to measure slopes on the surfaces, known as an inclinometer. Aaron Wangberg, a mathematician at Winona State, Wisconsin, invented and constructed the materials for use in a multi-variable calculus class.[3]

The course met for one hour on Mondays, Wednesdays, and Fridays and for two hours on Tuesdays and Thursdays. The double hours eased the integration of activities seamlessly into instruction, although the instructor frequently used interactive engagement techniques during the one-hour sessions as well. The session interpreted here occurred during a two-hour block on Day 4, Thursday, October 2, 2014.

In earlier class sessions, the students had learned to use power series expansions to estimate electric potentials in a variety of configurations. They soon would be learning how to use the gradient of an electrostatic potential in obtaining an expression for the associated electric field. They also soon would be learning by analogy about the use of the gradient in expressing mathematically the relation between the gravitational potential and gravitational field.

The instructor, Corinne Manogue, had several goals in mind for this class session. The subject matter goal of the episode interpreted here was helping students learn how to think about the physical meaning of a gradient. The gradient of a function of more than one variable provides information not only about the magnitude of the steepest change in the function at a point but also the direction in which that steepest change occurs.

This instructor also had several more general goals related to welcoming these students into the community of practicing physicists. In addition to helping students to become confident and competent in using advanced mathematics to describe physical phenomena, she wanted them to be able to work comfortably on ill-defined and multi-step problems. She also expected to shape their ways of speaking so that they could communicate their own ideas clearly and would be able to collaborate well with peers during activities in class as well as to work together productively on homework outside of class.

The authors video recorded their discussions of the video of this class. The instructor, Corinne, is a professor of physics who designed and has taught this course for many years. Emily van Zee is a science teacher educator, who in writing this narrative drew upon her own teaching experiences in laboratory-centered physics courses for prospective teachers[4] and her research in the tradition of ethnography of communication.[5]

Ethnographers of communication[6,7] examine cultural practices by interpreting what is said, where, when, by whom, for what purpose, in what way, and in what context. This interpretative narrative presents an example of an instructor initiating students into the culture of physics, specifically into the verbal and mathematical language that physicists use in describing changes in physical phenomena that can be represented as functions of two variables.

In the transcript below S: indicates what a student said. C: indicates what the instructor, Corinne, said during the class session. Corinne's comments while watching the video are indicated by quotation marks at the beginning of paragraphs and at the end of the last of a series of her statements quoted.

Introducing the Class Session

Before enrolling in this course, students had experienced the concept of 'steepness' in many contexts, from toddling up or down hills as young children, to calculating 'rise over run' in middle school, to interpreting slopes of linear functions in high school, to driving and/or skiing on steep inclines as young adults, and to encountering the formal definition of gradient in an advanced calculus course earlier in college. Thus this session was technically a review, although Corinne had found in previous years that the students rarely seemed to remember much about this topic.

The gradient of a function f of two variables (x, y) is defined, in rectangular coordinates, as $\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}\right)$ where $\frac{\partial f}{\partial x}$ represents the partial derivative of the function f with respect to x while holding y constant, $\frac{\partial f}{\partial y}$ represents the partial derivative of the function f with respect to y while holding x constant, and $\hat{x}$ and $\hat{y}$ represent unit vectors defining the x and y direction..

The symbol $\vec{\nabla}$, known as *del,* represents the vector operations that act on a scalar function f to obtain the gradient. Specified here in rectangular coordinates in two dimensions, *del* is written:
$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$

During this class session, the instructor and students considered the meaning of the gradient's component partial derivatives and unit vectors, its physical representation in the context of the innovative materials, and its use in describing physical phenomena. First we present the activity in which students used the innovative materials to explore the relation between different slopes in different directions. Then we present excerpts from a whole group discussion that helped students make sense of their explorations.

Using Clear Surfaces, Topo Maps, Grid Sheets and an Inclinometer to Explore the Physical Meaning of *Gradient*

This was the first day that the students were using innovative materials that provided physical representations of the mathematical entity known as the gradient. These innovative materials consisted of a set of six clear plastic surfaces, related topographical maps, sheets with coordinate grids, and a wooden tool made of dowel rods known as an *inclinometer* as shown in Figure 2.

Figure 2. Surface, topo map, and inclinometer.

The inventor of these tools, Aaron Wangberg, intended the surfaces to be tangible manifestations of functions of two variables. Each surface is about 10x10x5 inches and looks like a miniature mountain range with many peaks and valleys. The six versions of the clear surfaces are color-coded. Each surface has three dots, which represent particular points on the surface in which the students might be interested or at which the students can be given tasks to do.

The surfaces also come with several plastic-coated sheets that go underneath. One is a topographical map of the surface, known as a *topo map*. The three-dimensional surfaces can be used to represent mountainous terrains, or in analogy, the height of a point on the surface could represent the value of some other scalar such as temperature. In an analogy to a curve connecting points of equal elevation on a contour map of mountainous terrain, a contour line on these topo maps connects points of equal values of the function represented by the surface. If the surface represents temperature as a function of position (x, y), for example, a contour line on the topo map would represent a path of constant temperature, an isotherm. Corinne finds the topo map as exciting a representation by itself as the surfaces are. The ability to go back and forth between the topo map representation and the surface representation is particularly powerful.

There also are sheets that are rectangular coordinate grids or polar coordinate grids. The surfaces are clear so that when you set the surface on top of its topo map, or one of the coordinate grids, you can see below it and see the domain of the function.

The surfaces also come with a tool called an inclinometer, which allows the students to measure the slope at any point on the surface, in any direction. The inclinometer consists of a round dowel and a square dowel hinged, the two pieces being about a foot long. At the end of the square dowel is a little round level bubble.

The students put the inclinometer on the surface at the point at which they want to measure the slope and angle the round dowel so it is pointing along the direction in which they want to find the slope. Next they make the square arm level, using the little level bubble. The angle formed by the inclinometer represents the slope. Then they pick the inclinometer up without bending it further and lay it down on the rectangular grid, where they can determine the rise and the run and calculate the slope as shown in Figure 3. Because the inclinometer is big, the rise and run are both large numbers so that minimizes the uncertainty.

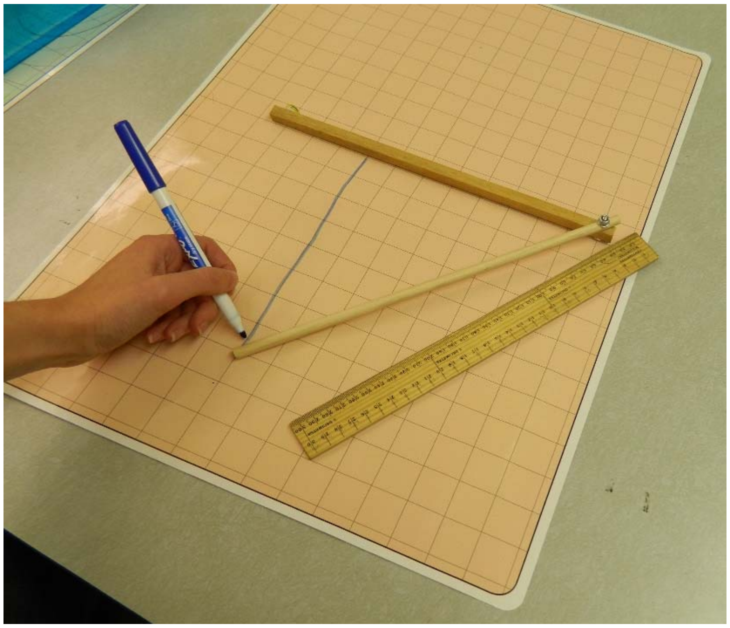

Figure 3. Use of inclinometer on a grid to estimate the slope

Inviting Students to Explore the Surfaces, Topo Maps, Grids, and Inclinometer

While watching the video, Corinne reflected upon her choice in selecting this particular class session to interpret in a narrative. She commented:

“When students are first learning about partial derivatives as opposed to ordinary derivatives of one variable, they are often not aware of all of the different options that they have. The students have a tendency to think of derivatives as an operation that you do to a function. It is a secondary idea that if you take that new function, called the derivative, and you evaluate that function at a point, the number you get is the slope of the tangent line of the original function, evaluated at that point.

“So the first idea (a derivative as an operation you do to a function) is totally algebraic and symbolic, and the second idea (the number you get is the slope of the tangent line of the original function, evaluated at a particular point) is primarily geometric but it has to be implemented numerically or symbolically.

“When we ask students to generalize that conceptual understanding to functions of two or more variables, then they not only have to deal with the question of, at which point they are going to evaluate the derivative, but also the idea that, at that point, they can find the slope in an infinitude of directions.

“What I did here in this class day was ask the students a very open ended question: find the derivative and I wanted them to generate the questions, as in “at which point?” and “in which direction?” This is a class where almost all of the students will already have had a multi-variable calculus class so the idea of partial derivatives and the gradient should be review.”

Corinne had initiated the small group activity in a very open-ended way. As she reflected on the video:

“I just handed the students the surfaces , rectangular girds, and inclinometers and asked them to measure the slope without giving them instructions about the inclinometer because I was curious whether or not it would be obvious to them how to use it. So I gave them a couple of minutes and it was pretty clear it wasn't obvious, so I stopped them and told them and we went on.”

Commenting to the Class about What Had Just Happened, Making a Connection to Their Future Lives as Physics Professionals, and Inviting the Students to Ask Questions

In class, however, Corinne first chose to comment upon the open-ended nature of her initial request and also to make a connection to what would be expected of the students later as professionals, not only to be able to figure out what to do on their own but also to know when and how to ask for help.
[00:05:56.26]

C. Ok. Let's pause for a minute now that you've had a chance to explore.
Again, this is what is going to happen to you in your work:
Your boss is going to say to you, "take this thing and take this thing and go mess with it"
and you're going to say, like, "hmmm, I don't know what you mean. All right.
So what you do is you go off and you try to figure it out for yourself for a little while
and then you ask for help, all right.
So, what are some of the questions that you now have. Raise your hand if you have questions.
[00:06:26.09]

Corinne typically interweaved such comments within on-going discourse as a way to provide both a broader context for the activity and an underlying explanation for why she was choosing to ask, say, or do whatever might be perceived as unexpected by the students. In this case, she had just asked them to do something without providing explicit directions with the result that they had spent several bewildering minutes not knowing quite what to do. Instead of now explicitly telling them what she had intended, she invited them to ask her questions, a speech act perhaps unprecedented in their experience as students previously taught primarily through lectures.

Coaching Students to Identify the Appropriate Questions to be Asking about Derivatives:

A student responded to this open invitation to express publically what was puzzling him:

S: What kind of derivative do you want?

Corinne re-voiced the question so all could hear, with emphasis on an aspect that needed refining:

C: What KIND of derivative? (pause) yes

Another student recognized the hint that Corinne was providing by her tone and, given her willingness to wait[8] for a response, offered an appropriate revision:

S. In which direction do you want the derivative?

Corinne again re-voiced the student's response so all could hear, again emphasizing with her tone a relevant aspect. She also provided a prompt to consider whether the two student offerings were the same or different:

C: In which DIRECTION do I want the derivative?
Is that the same as your question or different?

S: Somewhat the same.
C: ok.

Another student then articulated the important aspect that Corinne was hoping would emerge:

S. At what point do you want the derivative evaluated?

Corinne again re-voiced this student's response so all could hear, also again emphasizing the aspect she wanted all the students to be considering:

C: At what POINT do you want the derivative evaluated?

After several more student questions: "Are these the only tools you can use?" and "Are you measuring angles or distance?" Corinne provided the guidance she had been expecting would need to be offered, but now with the class as a whole paying close attention:

Summarizing in a Mini-lecture What the Students Need to be Thinking When Working with Derivatives

[00:07:35.04]
What I wanted you to realize is that, yes, if I ask for a derivative, the derivative,
the first thing you have to say is "at what point?"
because…your surfaces all have colored dots,
so pick one of those colored dots at which to find the derivative
S: (?)

Corinne then continued with a mini-lecture, crafted to alert the students to what they needed to be thinking about when working with derivatives:

C: All right, the next question is, in which direction do you want the derivative?
So remember, the derivatives are telling you about how one variable changes when you change the other one.
It's the ratio, you can think now about the derivative as the ratio of small changes.

Modeling the Process that a Derivative at a Point Represents

Next she walked over to a nearby table, reached for one of the surfaces, and reminded the students about what the surfaces represented, the way that a variable, such as temperature, was changing with position (x, y).
[00:08:22.12]
C: If I'm looking at this surface that I want to think about, how is the temperature changing as I take a step on this surface?

Then she introduced another physical representation of the process she was modeling, a small white board to represent an infinitesimal area on which she was envisioning measuring the temperature. As she took a step forward, she reached for a small white board and held it up:

C: so this is where, this is the square on which I measure the temperature
So if I move from this one point, a little infinitesimal, however small an infinitesimal is,
a little nearby point, right, a small enough nearby point,
as I move from one point to another in here
(picked up surface and whiteboard) and if I think about it, that I'm at this point,
then I want to know how much the temperature changes,
so I want a little delta T and I want a little delta step
but what I hope you're starting to realize now is that if I'm at this point on the surface
(put down surface and held up small white board)
I can step in lots of different directions. So you have to specify the direction.

In reflecting on the video, Corinne commented,

“I am picking this moment because I think it is the beginning of a rich interaction between me and several different students. So I'm dancing; they are telling me things about the derivative and I'm acting them out by taking steps in the classroom and I am also gesturing with the surfaces. So I am giving them several different representations.”

These representations included:

- the physical surfaces representing the variation of temperature with position (x, y),
- the white board representing an infinitesimal area at which to measure the temperature,
- the symbolic language (a little delta T, little delta s),
- and her stepping motions.

Jointly Constructing the Process that the Derivative at a Point Represents

Corinne then invited the students to participate in resolving the dilemma she was helping them to envision –in which direction of many possible should she step?

[00:09:19.16]
C: What would be some good choices for the direction in which to step?

A student responded appropriately:

S: A coordinate that you have, an “x” or a “y”.

And she affirmed his response:

C: Yes, direction, a coordinate direction.

Corinne pointed to a set of three wooden dowels, assembled to represent the rectangular coordinates, x, y, and z, that were hanging from the ceiling. These had figured prominently in several ways during previous class sessions.

Then she again affirmed the student’s suggestion:
C: So we've got an “x” and a “y”
and if you like, you can put your surface on top of a topo map, right?,
because it's got a grid on it.

As she picked up a map, she elaborated on the student’s suggestion:

C: so you've got an “x” and a “y” defined by the grid on your topo map if you like
so you could take the derivative in the x direction, you could take the derivative in the y direction

Then she paused, picked up a marker to write on the whiteboard, and rather than writing the next step herself, she probed the students for more information from their previous studies of the gradient:

Building a Comfortable Classroom Culture while Shaping Language

C: Are there, what are the symbols that you know for those derivatives?

However, she was surprised by what a student replied:
[00:09:58.01]
S: Del

and allowed her surprise to show in her tone:

C: Del? What's a del?
S: It's the little curvy

Her response was immediate and firm:
C: It's not.

She wrote an upside down triangle on the board ∇ and stated:
C: That's a del

The student persisted, referring to a ∂ Corinne had also written on the board:
S: Isn't the other one a del too?

And Corinne's response was again immediate and firm:

C: Nope

She pointed to ∂ and stated:

C: This one? It's just a curly d
S: Ok
00:10:15.20]
(students muted laughter)

While watching the video of this interchange, Corinne reflected,

"So I think this little interchange says something about the importance of comfortableness in the classroom. The student has the wrong name for a symbol that he knows how to write. And I've asked for the symbol without specifying how you write it or what you call it. So the student proposes the name *del* and I'm really surprised and so I say "del" with a you're-out-of-your-mind tone of voice to it but I keep him on the spot and I say "what's a del?" and he says, "it's the little curvy"

"He gets part of a statement out and I interrupt him with a very firm statement, "it's not" and then I write on the board a $\vec{\nabla}$ and say "that's a del" but the student is willing to come back with the statement "Isn't the other one a del too?" and I interrupt him again, with what could be quite a curt "Nope!" so I have now told him twice publically in front of the whole class that he's just plain wrong and after the "nope!" there is a little bit of, he talks again and I can't pick up what he says and so I think to draw on the board a curly d to ask him if this is what he means and I ask him, I do go back to him, I say "this one?" and so the flavor of the whole conversation is really technical and "you've got to get it right" - I'm not allowing him to get it wrong.

"Then I give the name for this symbol and I say "it's just a curly d," it's technical name is just a *curly d*, and he says "OK" and the class laughs because, after all this emphasis on being technical, it has this silly name.

"But I think it is really important that this student is willing to challenge me a couple of times and insist on getting the answer to the question he wants and the tone of this banter back and forth between him and me ends up being light and humorous. And that's a signal to me that a comfortable classroom culture is existing with that student.

"I think it also points to why later there is a more substantive conversation between students."

While watching the video, Corinne also noted,

"There's an interesting interchange between the two students visible on the corner of the video frame: The neighbor of the person who was talking says 'I always thought that was a del too.'

There's quite a bit of conversation among this group with each other."

What is known as *sidetalk* often occurs among members of small groups during whole group discussions in this course. Such conversations are often revealing about what students know and can deepen our understanding about the issues they face in making sense out of our instructional efforts. In addition to video recording the whole group discussion, we had video recorded three of the six small groups so had access to some of these conversations during this session.

Continuing to Shape Language while Jointly Reconstructing What Students Need to Remember

Corinne continued engaging students in jointly reconstructing the information she wanted them to remember:

[00:10:19.19]
C: ok. It's got that technical name, curly d, (pointed to board) so it's curly d what? (turned to class)
A student responded using the newly defined 'technical' vocabulary:

S: Curly d I guess f, curly d x

However, now Corinne chose to move toward standard nomenclature, but did so by again explicitly welcoming the students into the community they were joining:

C: We're going to be physicists. We're going to start calling the functions by their name; it's a temperature, so curly d T d x

After writing $\frac{\partial T}{\partial x}$ on the board, she turned to the class and asked:
C: Does this have, does this kind of derivative have a name?
[00:10:47.26]

Several students responded:

Ss: This is a partial derivative

Prompting Emergence of the Next Issue Needing Discussion

Corinne offered an affirmation, elaboration, and probe to prompt the next issue needing discussion:

C: This is the partial derivative (writes on board "partial derivative")
[00:10:51.05]
Ok. and that means a step in the x direction. (writes on board $\frac{\partial T}{\partial y}$)
We would also have a partial derivative of T with respect to y (turns to class)
What else do we have?

(pause)
What other directions might we step?

A student offered the response she was anticipating, that if dx and dy have been discussed, dz also would be relevant.
S: d z(?)

Corinne wrote "dz" on the board and asked the question she wanted them to ponder:
C: dz! How would we measure "dT dz"?

Continuing to Shape Language while Encouraging Student Participation

The student's response raised another opportunity for shaping language that Corinne had been addressing during previous class sessions:
S: It's 3d
C: "It is 3 d". What is 3d? No, no pronouns, which 'it'?
[00:11:22.01]

While watching the video, Corinne explained:

"I have established a classroom norm that the students are not allowed to use pronouns because I've found that the use of a pronoun often hides from the speakers themselves which thing they are referring to and it certainly hides from me and from the rest of the class which thing they are referring to.

"In this particular case, I ask the student how you would measure $\frac{\partial T}{\partial z}$, because I know that the temperature is only a function of x and y and so $\frac{\partial T}{\partial z}$ doesn't exist.

"The student responds with something about 'it' being 3 dimensional and I suspect that the student means that the surface is 3 dimensional, which it is, but the surface is intended to be a 3 dimensional representation of a function of two variables, where the vertical direction on the surface is meant to be the value of the function, this is, in this case, temperature.

"So the student is almost certainly manifesting a standard or a common misunderstanding about the surface. By calling the student's and class's attention to what the "it" is supposed to be referring to, I'm trying to generate a discussion about this misunderstanding.

"Instead of addressing this issue explicitly, the class goes off onto some comments about contour maps. It may be that the students who are talking about contour map are trying to point out the two dimensional nature of the domain of the function. So I'm trying in the next little bit I'm trying to figure out what the students are trying to say and echo it and/or interpret what they're saying for the rest of the class.

S1:
S2:

S1:

C: Is that a contour map? Or is the thing underneath? OK, I would call the thing underneath a contour map.

Emphasizing an Issue that Typically Causes Difficulties:

Corinne emphasized the issue by picking up a whiteboard to represent the infinitesimal flat surface on which they are envisioning the temperature being measured:

C: Right, so what we're doing is we're measuring the temperature on a 2-D surface
[00:11:42.19]
so this temperature is a function of two variables, x and y

She wrote T(x,y) on the board and interpreted what that meant:
C: When I give you that plot, I'm saying that I measured the temperature here and it was 72.1 and here it was 72.0 and here it was 72.2 or something, I did all of those
and then I made a plot where the vertical direction was the value of the temperature.

When watching the video, Corinne expressed surprise at her own language:

“I find it interesting that I was referring to the surface and I called it a plot. I'm talking about physically handing them a surface and I'm calling it a plot. It's telling me something about myself, that I think of it as a graph, that the physical surface to me is a graph, and I'm using language unconsciously that is hopefully conveying that understanding to the students. I'm happy that I did this, I'm just surprised I did.

[00:12:11.04]
Then she emphasized this aspect of interpreting gradient with the surfaces, that the vertical direction represents the value of the function and not a vertical distance z. She emphasized this both verbally and by picking up and pointing to one of the surfaces.

C: All right? So not z anymore, not a rectangular, not a distance
[00:12:17.17]
so, hand me your surface, so this direction up here, is T and not z.
All right. So I can't use, I mean I have both T and z,
I don't know how to do this derivative (pointed on board to $\frac{\partial T}{\partial z}$)
Am I making sense to you?
S: Yeah
C: Ok

After checking with the students, Corinne continued her mini-lecture, both explaining the reason for a previous activity and catching herself making the same mistake she was cautioning against the students making:

[00:12:36.17]

C: So part of the reason I was trying to have you do level curves and stuff the other day
is to exactly try to get at this confusion about 3d graphs
whether z is an anything or isn't an anything
or is, so now I mean temperature.
Is everybody now real clear about T of x, y, z?
Sorry, this is not,... what I meant, not what I said. T of x and y? All right.

Considering What Other Derivatives Might be Relevant, such as a Combination of Derivatives

After handing the surface back to the student, Corinne launched the next aspect to be considered:
C: Ok. so what other derivatives might I mean?

S: Combination
[00:13:09.28]

C: A combination! What combination might I want?

S: d T d x plus d T d y

C wrote on board $\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}\right)$
Corinne acknowledged the student's suggestion and then asked for what that would mean geometrically:

C: Once you've got a couple of derivatives, these are just numbers. I can add them
[00:13:28.03]
What would that mean geometrically to add them?

S: Some other direction (?) That would be like say 45 degrees between x and y

C: "That" - no pronouns

Ss: (giggle)

S: Uh, (pause) duh, the sum of "d T d x" plus "d T d y" would be the derivative in the direction of 45 degrees or pi over four between the x and y axis.

In watching the video, Corinne commented on the effect of having pushed for appropriate language, " The waffle words here are things like 'um', thinking words, he's taking his time now to make a precise statement."
C: ok.

Working with a Student's Unexpected Suggestion

If one opens up the conversation for student participation, one often hears something different from what one is anticipating. In this case, Corinne chose to act out what the student was suggesting to clarify what this suggestion was:

C: So if I say that when I step in the x direction

She acted this out by stepping to one side, stopping, and stepping to the opposite side, in directions that matched the coordinates hanging from ceiling, and continued with a specific example:

C: If I step in the x direction, a small step, right, the temperature changes 2 degrees (pointed to board) per foot
[00:14:13.03]
All right. And if I step in the y direction, it changes only 1 degree per foot
[00:14:17.23]
Are you telling me that if I step in the x plus y direction, (steps on the diagonal),
it is going to change 3 degrees per foot?
S: (?)
C: 2 degrees this way, 1 degree this way (acts this out), that's what you (points to board)
S: I take it back.

Affirming a Student's Contribution that is Only Partially Correct

Corinne had acted out the student's suggestion to make clear why this would not work but affirmed his effort and partial progress:

C: You take it back, but don't totally take it back
[00:14:39.04]
There is a core of something here which is really really important

Building upon This and Additional Student Suggestions

Another student raised the idea of using a dot product and Corinne worked with the students to refine this suggestion

Another S: you dot it with your vector, which direction, from
[00:14:46.13]
C: dot which "it"
S: to dot the x and y directions with your vector direction

C: Ok so now you are trying to invoke the dot product.
The dot product acts on
(pause)
S: a vector

C: A vector?

S: Two (gestures by raising two fingers)

Corinne reiterated this contribution and encouraged the student to continue:

C: The dot product acts on TWO vectors <so> and it takes two vectors to a scalar
Ok. so keep going.

S: so if you make your x and your y derivatives a vector and dot that with the direction that you're trying to go, which (?) a vector and then it gives you the derivative in that direction

Corinne acknowledged this complex suggestion, asked the class if they had heard it, and began interpreting what he said:

C: Ok. Did you all hear what he said?
C: He said, take these two derivatives and make them a vector
That's a very interesting operation. All right.
Take two derivatives and make them a vector.

Then she invited the class to participate in the thinking:
C: How do I do that?

S: You just call it a vector
Ss: (laughter)
C: No "it's". You call what a vector?
So the mathematicians did something (turns to write on board)
the mathematicians did something really weird to you
(laughter)
They said (writes on board $\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)$) take the two numbers and make them a vector,
by just writing them right next to each other
C: (?) (turns to watching mathematician and laughs)
M: (?) You use it all the time, it's not really weird.
C: (laughs) I learned it from mathematicians
(turns to board)
All right. I can tell we're going to have a tennis match in this class now. (laughter)

Returning to the Use of Language to Introduce the Term *Gradient*

Corinne elicited from the students the language she wanted them to use:

C: But I just told you something about making vectors, are we going to use this language?
[00:16:34.25]
so, what language do we use, to make vectors?
S: (x hat?)
C: x hat, we'll write them with an x hat

C: Who said x hat?
S: (raises hand)
C: You said x hat (writes on board $\frac{\partial T}{\partial x}$)
d T d x
S: Times x hat
C: Times x hat? Times x hat, what's times mean?
It means write it next to each other (writes $\hat{x}$ to read $\frac{\partial T}{\partial x}\hat{x}$)
Alright? Because
this a number and that's a unit vector
and
S: (?)
C: dy y hat
(turns to class)

She invited the students to reflect on the entity they had just assembled together:

C: All right. So now, here's this strange object, because it's a vector and a derivative
It's a really bizarre thing, so does that bother you?
Yes? question or a comment? or I'm thinking hard. All right. Yes

and gave it a name:

S: That's just the gradient (?)
C: Has a name. So this has a name! Ok. This is a ***gradient.***
(writes on board "Gradient")

So it's the combination you were looking for
(looks to earlier student who suggested adding them)
but don't forget the x hat and the y hat.
[00:17:45.15]
All right. You don't want to just add them but somehow (points to board at $\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}$).
While watching the video, Corinne noted, "Then I go on to pose a new prompt, we've developed the idea of a gradient, then I ask, how can it be both a derivative and a vector?"

Reflecting on the Video

Near the end of our debriefing session, Corinne reflected upon the important aspects of this discussion:

"I think there are three main things:

"One is trying to get students to make precise statements about what they are talking about and using the ban on pronouns to force that. Looks like that came up a number of different times.

"Another is that there are several places, but one in particular, where a student says you can add derivatives, where a student talks about an incomplete but partially correct understanding of the content under discussion. So I think this is a really good example of how to take that idea seriously and build on it rather than saying no and discarding it. And eventually other students have to come in with additional aspects of the correct idea but the class builds on the final answer jointly in this way. I feel strongly that that models for the students how to keep pursuing an idea rather than giving up, even in their own personal problem-solving.

"I think that there is some really interesting interplay between the students and there's also interesting to me how different students take up part of the story.

"The fourth is demonstrating all the different representations that are useful to get at the idea of a gradient and things like the fact that there are axes in the ceiling, the algebraic representations on the board, me physically dancing around acting out the motions, the surfaces and topo maps.

"I think I pose a real open-ended question (what other derivatives can you find at that point?) and a student mentioned a combination of the two partial derivatives. Because this was review I expected a student to go directly to either directional derivative or gradient but this student just added the vectors. That was unexpected. Part of it is respecting student partial answers and knowing how to run with them. I think another aspect is knowing how far to push a student with a partial answer to say more, to articulate more, and when and how to invite the rest of the class back into the conversation."

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