Activity 2: Solution for electric potential due to a ring

Find the electrostatic potential in all space due to a ring with total charge Q and radius R

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i|} \tag{1}$$

For a ring of charge this becomes

$$V(\vec{\boldsymbol{r}}) = \int_{\text{ring}} \frac{1}{4\pi\epsilon_0} \frac{\lambda(\vec{\boldsymbol{r}}')\,|d\vec{\boldsymbol{r}}'|}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} \tag{2}$$

where $\vec{\boldsymbol{r}}$ denotes the position in space at which the potential is measured and $\vec{\boldsymbol{r}}'$ denotes the position of the charge.

In cylindrical coordinates, $|d\vec{\boldsymbol{r}}'| = R\,d\phi'$, where R is the radius of the ring. Thus,

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda(\vec{\boldsymbol{r}}')\,R\,d\phi'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} \tag{3}$$

Assuming constant linear charge density for a ring with charge Q and radius R, $\lambda(\vec{\boldsymbol{r}}') = \frac{Q}{2\pi R}$ Thus,

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} \tag{4}$$

Since $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}'$ are not necessarily in the same direction, we cannot simply leave $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$ in curvilinear coordinates and integrate directly. One solution to this problem is to rewrite $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$ in cartesian coordinates

$$|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \tag{5}$$

Setting the ring in the x, y plane with the center at the origin and then rewriting in cylindrical coordinates results in

$$|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'| = \sqrt{(r\cos\phi - R\cos\phi')^2 + (r\sin\phi - R\sin\phi')^2 + (z-0)^2} \tag{6}$$

Which simplifies to

$$|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'| = \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} \tag{7}$$

Substituting into Eq. 4 results in the elliptic integral

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{8}$$

1 The z axis

For points on the z axis, $r = 0$ and the integral simplifies to

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{\sqrt{R^2 + z^2}} \tag{9}$$

And thus

$$V(r,\phi,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2+z^2}} \tag{10}$$

1.1 Power series expansions for z axis

To create the power series expansion for $|z| << R$, factor out R from the denominator

$$V(r,\phi,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{R}\frac{1}{\sqrt{1+\frac{z^2}{R^2}}} \tag{11}$$

Using the power series $(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + ...$ results in

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{R}\left(1 - \frac{1}{2}\frac{z^2}{R^2} + \frac{3}{8}\frac{z^4}{R^4} + ...\right) \tag{12}$$

The power series expansion for $z >> R$ is

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2} + \frac{3}{8}\frac{R^4}{z^4} + ...\right) \tag{13}$$

2 The x axis

For points on the x axis, $z = 0$ and $\phi = 0$, so the integral simplifies to

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{\sqrt{r^2 - 2rR\cos\phi' + R^2}} \tag{14}$$

Which can be rewritten as

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}(r^2 - 2rR\cos\phi' + R^2)^{-1/2}d\phi' \tag{15}$$

In this case the power series expansion can be done before integration and then the power series can be integrated. For $x >> R$, factor out an $1/r$ to obtain

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{1}{r}\left(1 - \frac{2R}{r}\cos\phi' + \frac{R^2}{r^2}\right)^{-1/2}d\phi' \tag{16}$$

Let $\epsilon = -\frac{2R}{r}\cos\phi' + \frac{R^2}{r^2}$

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{r}\int_0^{2\pi}(1+\epsilon)^{-1/2}d\phi' \tag{17}$$

The power series expansion now yields

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{r}\int_0^{2\pi}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{15}{48}\epsilon^3 + ...\right)d\phi' \tag{18}$$

Substituting $-\frac{2R}{r}\cos\phi' + \frac{R^2}{r^2}$ for ϵ results in the integrand

$$1 + \left(-\frac{1}{2}\right)\left(-\frac{2R}{r}\cos\phi' + \frac{R^2}{r^2}\right) + \left(\frac{3}{8}\right)\left(\frac{4R^2}{r^2}\cos^2\phi' - \frac{4R^3}{r^3}\cos\phi' + \frac{R^4}{r^4}\right) \tag{19}$$

$$+ \left(-\frac{15}{48}\right)\left(-\frac{8R^3}{r^3}\cos^3\phi' + \frac{8R^4}{r^4}\cos^2\phi' - \frac{4R^5}{r^5}\cos\phi' + \frac{R^6}{r^6}\right) + ... \tag{20}$$

Adding like terms and getting rid of any powers greater than third-order in r yields

$$1 + \frac{R}{r}\cos\phi' - \frac{R^2}{2r^2} + \frac{3R^2}{2r^2}\cos^2\phi' - \frac{3R^3}{2r^3}\cos\phi' + \frac{5R^3}{2r^3}\cos^3\phi' + ... \tag{21}$$

Using this power series and performing the integral results in the first two non-zero terms for the potential

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{r}\left(2\pi + \frac{\pi}{2}\frac{R^2}{r^2} + ...\right) \tag{22}$$

Which can be simplified to

$$V(r,\phi,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\left(1 + \frac{1}{4}\frac{R^2}{r^2} + ...\right) \tag{23}$$

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