Group Activity 7: The Wire

I Essentials

(a) Main ideas

- Calculating (vector) line integrals.
- Use what you know!

(b) Prerequisites

- Familiarity with $d\vec{\boldsymbol{r}}$.
- Familiarity with "Use what you know" strategy.

(c) Warmup

This activity should be preceded by a short lecture on (vector) line integrals, which emphasizes that $\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ represents chopping up the curve into small pieces. Integrals are sums; in this case, one is adding up the component of $\vec{\boldsymbol{F}}$ parallel to the curve times the length of each piece.

A good warmup problem is §18.2:6 in MHG [3].

(d) Props

- whiteboards and pens

(e) Wrapup

- Emphasize that students must express everything in terms of a single variable prior to integration.
- Point out that in polar coordinates (and basis vectors)

$$\vec{\boldsymbol{B}} = \frac{\mu_0 I}{2\pi} \frac{\hat{\boldsymbol{\phi}}}{r}$$

  so that using $d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$ quickly yields $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$ along a circular arc ($\frac{\mu_0 I}{2\pi}\,d\phi$) or a radial line (0), respectively.

II Details

(a) In the Classroom

- Sketching the vector field takes some students a long time. If time is short, have them do this before class.
- Students who have not had physics don't know which way the current goes; they may need to be told about the right-hand rule.
- Some students may confuse the wire with the paths of integration.
- Students working in rectangular coordinates often get lost in the algebra of Question 2b. Make sure that nobody gets stuck here.
- Students who calculate $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = \frac{dy}{x}$ on a circle need to be reminded that at the end of the day a line integral must be expressed in terms of a single variable.
- Some students will be surprised when they calculate $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = 0$ for radial lines. They should be encouraged to think about the directions of $\vec{\boldsymbol{B}}$ and $d\vec{\boldsymbol{r}}$.
- Most students will either write everything in terms of x or y or switch to polar coordinates. We discuss each of these in turn.
  - This problem cries out for polar coordinates. Along a circular arc, $r = a$ yields $x = a\cos\phi$, $y = a\sin\phi$, so that $d\vec{\boldsymbol{r}} = -a\sin\phi\, d\phi\, \hat{\boldsymbol{\imath}} + a\cos\phi\, d\phi\, \hat{\boldsymbol{\jmath}}$, from which one gets $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = \frac{\mu_0 I}{2\pi}\, d\phi$.
  - Students who fail to switch to polar coordinates can take the differential of both sides of the equation $x^2 + y^2 = a^2$, yielding $x\,dx + y\,dy = 0$, which can be solved for dx (or dy) and inserted into the fundamental formula $d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}$. Taking the dot product then yields, $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = \frac{\mu_0 I}{2\pi}\frac{dy}{x}$. Students may get stuck here, not realizing that they need to write x in terms of y. The resulting integral cries out for a trig substitution — which is really just switching to polar coordinates.
- In either case, sketching $\vec{\boldsymbol{B}}$ should convince students that $\vec{\boldsymbol{B}}$ is tangent to the circular arcs, hence orthogonal to radial lines. Thus, along such lines, $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = 0$; no calculation is necessary. (This calculation is straightforward even in rectangular coordinates.)

- Watch out for folks who go from $r^2 = x^2 + y^2$ to $d\vec{\boldsymbol{r}} = 2x\,dx\,\hat{\boldsymbol{\imath}} + 2y\,dy\,\hat{\boldsymbol{\jmath}}$.
- Working in rectangular coordinates leads to an integral of the form $\displaystyle\int -\frac{dx}{y}$, with $y = \sqrt{r^2 - x^2}$. Maple integrates this to $-\tan^{-1}\left(\dfrac{x}{y}\right)$, which many students will not recognize as the polar angle ϕ. If $r = 1$, Maple instead integrates this to $-\sin^{-1} x$; same problem. One calculator (the TI-89?) appears to use arcsin in both cases.

(b) Subsidiary ideas

- Independence of path.

(c) Homework

- Any vector line integral for which the path is given geometrically, that is, without an explicit parameterization.

(d) Essay questions

- Discuss when $\displaystyle\int_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$ around a closed curve will or will not be zero.

(e) Enrichment

- This activity leads naturally into a discussion of path independence.
- Point out that $\vec{\boldsymbol{B}} \sim \vec{\nabla}\phi$ everywhere (except the origin), but that $\vec{\boldsymbol{B}}$ is only conservative on domains where ϕ is single-valued.
- Discuss *winding number*, perhaps pointing out that $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$ is proportional to $d\phi$ along *any* curve.
- Discuss *Ampère's Law*, which says that $\displaystyle\int_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$ is (μ_0 times) the current flowing *through* C (in the z direction).