Group Activity 4: The Hill

I Essentials

(a) Main ideas

- Reinforce the geometric definition of the gradient.
- Differences between 2-d and 3-d representations of hills.
- Emphasize that the gradient lives in the domain, not on the graph.

(b) Prerequisites

- The ability to compute the gradient of a function.
- The geometric interpretation of the gradient, that is, that the direction of the gradient of a function gives the direction of greatest increase of the function, while the magnitude gives the rate of increase in that direction.

(c) Warmup

- It takes the students surprisingly long to draw the topo map themselves. If you value this skill, have them draw it beforehand.

(d) Props

- whiteboards and pens
- any topographic map
- hill transparency (master on page 92)
- blank transparencies and pens

(e) Wrapup

- Have each group draw their gradient vector on the board on a single topo map. This is a good place to introduce vector fields.
- Ask students to stand up and point in the direction of the the gradient. Many will point "uphill", thinking that the gradient is 3-dimensional even though their computed answer does not contain $\hat{\boldsymbol{k}}$. Discuss this!

II Details

It's tempting to use a hill as a nice geometric example of a function of two variables. However, doing so opens a can of worms. In examples like this, when the function has dimensions of length, students are confused as to whether the gradient is 2-dimensional or 3-dimensional; see Chapter 10. In most applications, involving physical quantities such as temperature, this confusion does not arise. If you want to use hills as an important example, then it's best to confront this confusions head-on; this lab is a good way to do so, although this requires fairly sophisticated geometric reasoning. If you choose to restrict to other applications, you may prefer to skip this lab.

(a) In the Classroom

- You may wish to have students work on the points on the axes (**E** or **F**) first, then try another point.
- If you are not using an overhead projector, draw the topo map on the board while students are working on this activity so it's available for the wrapup.
- If you are using an overhead, put a blank transparency over the hill master and have students draw on that. If your classroom has white-boards rather than blackboards, you can also project the transparency directly onto the board, then have students draw on the board.
- In the last question, groups may get the scale wrong. If the horizontal part of their answer is a unit vector, then they should use the magnitude of the gradient as the z-component. But if the horizontal part of their answer is the gradient, then they must scale the z-component appropriately.

(b) Subsidiary ideas

- The gradient is perpendicular to level curves.
- Is the gradient in this activity 2-dimensional or 3-dimensional?
- Topo maps vs. graphs.
- Units!

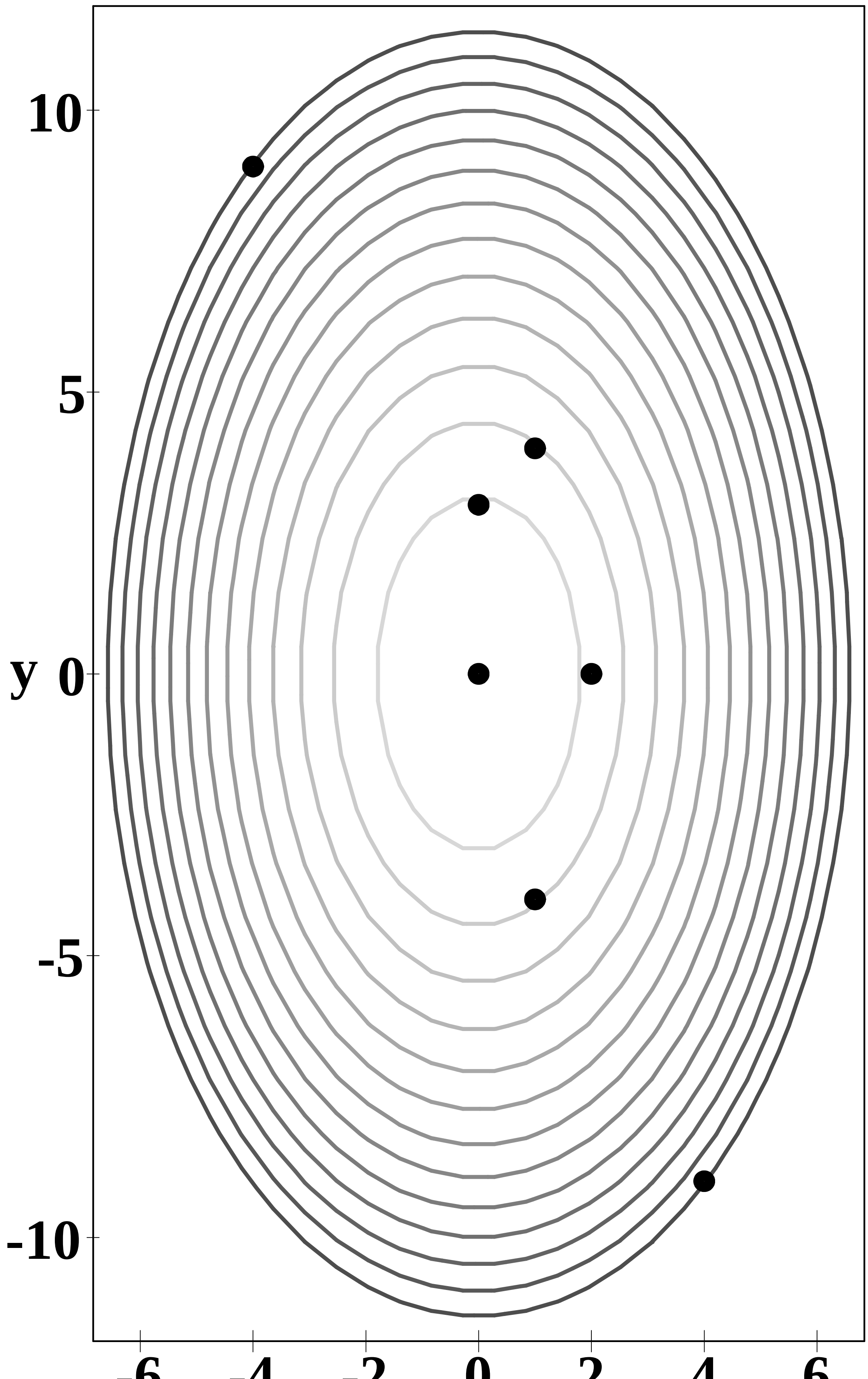
(c) Homework

Consider a valley whose height h in meters is given by $h = \frac{x^2}{10} + \frac{y^2}{10}$, with x and y (and 10!) in meters. Suppose you are hiking through this valley on a trail given by $x = 3t$, $y = 2t^2$, with t in seconds (and where "3" and "2" have appropriate units). How fast are you climbing *per meter* along the trail when $t = 1$? How fast are you climbing *per second* when $t = 1$.

(d) Essay questions (none yet)

(e) Enrichment

- What is the length of the vector in the last problem. What does it mean? Do the units need to be the same in each term?
- Discuss which way you should go to get to the top of the hill the fastest. What does this mean? The shortest path (geodesic)? The one with the largest *average* steepness? The *smallest* average steepness?
  As one student put it, the answer depends not only on the shape of the hill, but also on the shape of the hiker!
- Students may find it easier to visualize what it means to go *down* as steeply as possible. If you empty your canteen on the ground, which way would the water go? If you were skiing, which way would you go?
- Why is it that on a real hill, turning exactly $90°$ from the steepest direction always keeps you at constant height?
- Regard the hill as a level surface of the function of three variables $z - h(x, y)$. What is the gradient of this function? What does it mean geometrically? How is it related to the gradient of h and to the 3-dimensional vector found in the last problem?
  (This is a good problem for honors students.)
- Point out that if the drawing is not to scale — as produced for instance by the default settings for both Maple and *Mathematica* — then the gradient will not appear to be perpendicular to the level curves!
- Directional derivatives are rates of change with respect to "distance traveled". It is important to realize that the steepness of the hill is the directional derivative with respect to distance traveled *in the topo map* (the "run"), not physical distance traveled on the hill.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