Zeeman perturbation matrices in the coupled basis

The Zeeman effect occurs when an external magnetic field is applied to an atom. The system we wish to study is the hydrogen atom in the $2p$ state. To do the Zeeman effect properly, we must include the electron spin, but we can safely neglect the proton spin. For this problem, we assume that the magnetic field is weak (i.e., smaller than the fine structure), meaning that we must include the fine structure in the zeroth-order Hamiltonian and treat the Zeeman effect as a perturbation. The Zeeman Hamiltonian is

$$H'_Z = \frac{\mu_B B}{\hbar}\left(g_\ell L_z + g_e S_z\right) .$$

The fine structure is diagonal in the coupled basis, while the Zeeman perturbation is diagonal in the uncoupled basis. But we must use the coupled basis, because perturbation theory requires us to use the zeroth-order basis. So we must understand both bases for this problem. (see p. 397)

Small white board questions:

1) How many states are there in the $2p$ manifold (subspace)? What are those states in the uncoupled basis? Hint: what are the relevant quantum numbers in the uncoupled basis for the $2p$ manifold.

$$n=2;\ \ell=1;\ s=\tfrac{1}{2};\ m_\ell=1,0,-1;\ m_s=\tfrac{1}{2},-\tfrac{1}{2}\quad\Rightarrow\quad 6\ \textit{states}$$

$$\left|n\ell s m_\ell m_s\right\rangle=\left|\ell s m_\ell m_s\right\rangle=\left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle,\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle,\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle,\left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle,\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle,\left|1\tfrac{1}{2},-1\tfrac{-1}{2}\right\rangle$$

2) What are the states in the coupled basis? Same hint.

$$n=2;\ \ell=1;\ s=\tfrac{1}{2};\ j=\tfrac{3}{2},\tfrac{1}{2};\ m_j=\tfrac{3}{2},\tfrac{1}{2}-\tfrac{1}{2},\tfrac{-3}{2}\quad\Rightarrow\quad 6\ \textit{states}$$

$$\left|n\ell s j m_j\right\rangle=\left|j m_j\right\rangle=\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle,\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle,\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle,\left|\tfrac{3}{2}\tfrac{-3}{2}\right\rangle,\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle,\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle$$

3) Write down the matrix for L_z in the uncoupled basis, by inspection. Hint: use the eigenvalue equation. Hint: $L_z\left|\ell s m_\ell m_s\right\rangle = m_\ell\hbar\left|\ell s m_\ell m_s\right\rangle$.

4) Write down the matrix for S_z in the uncoupled basis, by inspection. Hint: use the eigenvalue equation. Hint: $S_z\left|\ell s m_\ell m_s\right\rangle = m_s\hbar\left|\ell s m_\ell m_s\right\rangle$.

5) Use the handout of Clebsch-Gordan coefficients to write down the coupled states $\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle,\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle$ in terms of the uncoupled states.

$$\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle=\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle+\tfrac{1}{\sqrt{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle=\tfrac{1}{\sqrt{3}}\left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle-\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle$$

6) Find the matrix element $\left\langle\tfrac{1}{2}\tfrac{-1}{2}\right|S_z\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle$. Put it into the matrix handout.

Large white board activities:

7) Find the matrix representation of the orbital angular momentum component operator L_z in the coupled basis.

8) Find the matrix representation of the electron spin component operator S_z in the coupled basis.

9) Find the matrix representation of the total electron angular momentum component operator J_z in the coupled basis. (either by inspection or by adding L_z and S_z)

More answers, hints:

3,4) See Mathematica printout

5) Use Clebsch-Gordan definition

$$\left| jm_j \right\rangle = \sum_{m_\ell m_s} \left| \ell s m_\ell m_s \right\rangle \left\langle \ell s m_\ell m_s \middle| jm_j \right\rangle$$

6)

$$\begin{aligned}
\left\langle \tfrac{1}{2}\,\tfrac{-1}{2} \right| S_z \left| \tfrac{3}{2}\,\tfrac{-1}{2} \right\rangle &= \left(\tfrac{1}{\sqrt{3}} \left\langle 1\,\tfrac{1}{2}\,0\,\tfrac{-1}{2} \right| - \sqrt{\tfrac{2}{3}} \left\langle 1\,\tfrac{1}{2}\,,-1\,\tfrac{1}{2} \right| \right) S_z \left(\sqrt{\tfrac{2}{3}} \left| 1\,\tfrac{1}{2}\,0\,\tfrac{-1}{2} \right\rangle + \tfrac{1}{\sqrt{3}} \left| 1\,\tfrac{1}{2}\,,-1\,\tfrac{1}{2} \right\rangle \right) \\
&= \tfrac{\sqrt{2}}{3} \left\langle 1\,\tfrac{1}{2}\,0\,\tfrac{-1}{2} \right| S_z \left| 1\,\tfrac{1}{2}\,0\,\tfrac{-1}{2} \right\rangle - \tfrac{\sqrt{2}}{3} \left\langle 1\,\tfrac{1}{2}\,,-1\,\tfrac{1}{2} \right| S_z \left| 1\,\tfrac{1}{2}\,,-1\,\tfrac{1}{2} \right\rangle \\
&= \tfrac{\sqrt{2}}{3} \left(\tfrac{-1}{2} \hbar \right) - \tfrac{\sqrt{2}}{3} \left(\tfrac{1}{2} \hbar \right) = -\tfrac{\sqrt{2}}{3} \hbar
\end{aligned}$$

7,8,9) See Mathematica printout