Research-based Quantum Instruction: Paradigms and Tutorials

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A growing body of research-based instructional materials for quantum mechanics has been developed in recent years. Despite a common grounding in the research literature on student ideas about quantum mechanics, there are some major differences between the various sets of instructional materials. In this article, we examine the major institutional considerations that influenced the development of two comprehensive quantum mechanics curricula: *Paradigms in Physics* (the junior-level physics courses at Oregon State University) and *Tutorials in Physics: Quantum Mechanics* (a set of supplementary worksheets designed at the University of Washington). The institutional considerations that we consider vary in nature: some are philosophical or theoretical commitments about teaching and learning, while some are practical structures determined in part by the local institutional environments. We then use these instructional considerations as a lens to explore example activities from each curriculum and to highlight prominent differences between them, along with the underlying reasons for those differences. The theoretical commitments of the *Paradigms* were strong enough to drive changes to the practical structures, whereas the practical structures of the *Tutorials* constrained what theoretical commitments could reasonably be adopted. Partially as a result of this large-scale difference, we find that each curriculum prioritizes different theoretical commitments about how best to promote student understanding of quantum mechanics. We discuss instances of both alignment and tension between the theoretical commitments of the two curricula and their impact on the instructional materials.

I. INTRODUCTION

Quantum mechanics is an essential part of upper-level physics instruction. At the undergraduate level, quantum mechanics courses are part of the physics core, forming a foundation for both future coursework and research. Students tend to be excited to study quantum mechanics, which is typically only discussed briefly in high school or introductory physics courses. However, quantum courses can be particularly challenging: they present students with physical behaviors that run counter to students' classical intuitions, and they typically require the use of advanced mathematical techniques.

Over the last 25 years, the physics education research community has accumulated a large body of research on student understanding of quantum mechanics. The research on student ideas about different topics has been particularly broad, including wave properties of matter [1], probability [2], quantum tunneling [3], time dependence [4–6], measurements [7, 8], angular momentum [9, 10], and perturbation theory [11]. This research has been supported by results from the development of several formal conceptual assessments [5, 12–15]. It has also influenced the development of instructional material aimed at improving student learning of quantum mechanics (see, for example, Refs. [9, 16–25]). Each set of material attempts to improve student understanding in different ways and using different pedagogical strategies, many of which have been inherited from the more extensive body of literature on teaching, and more specifically on teaching introductory physics. There has also been research assessing the effectiveness of such curricula [26–33].

In this article, we examine two of the most comprehensive sets of instructional material for teaching upper-level quantum mechanics: *Paradigms in Physics* and *Tutorials in Physics: Quantum Mechanics*. The first curriculum, the *Paradigms in Physics* [24, 34, 35], is a reimagined sequence of upper-division courses that makes extensive use of a diverse set of strategies for active student engagement and takes a non-traditional approach to the sequencing of physics content. We focus in this paper primarily on the QM aspects of the *Paradigms* program—more detail about some of the non-quantum aspects of the curriculum development may be found in Ref. [36]. The second curriculum, *Tutorials in Physics: Quantum Mechanics* [23], is a set of supplementary worksheets in the style of *Tutorials in Introductory Physics* [37] that is intended to support conceptual understanding in a small-group problem-solving setting. Throughout this article, we use the standalone terms *Paradigms* and *Tutorials* to refer to content that is related to *quantum mechanics*, and not to refer to the non-quantum *Paradigms* or to *Tutorials* in either introductory or other advanced areas of physics.

Each of these two sets of material leverage both the research literature about students' ideas and many years of accumulated pedagogical content knowledge [38, 39] in quantum mechanics, though they do so in different ways. In particular, developmental decisions are informed by *instructional considerations* that are different for the two curricula. We consider two different kinds of instructional considerations: *theoretical commitments*, which arise from instructional philosophies and theories of learning, and *practical structures*, which emerge from the institutional and structural environment of a curriculum. We acknowledge that these two kinds of considerations are not necessarily distinct, and we

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have found that they often influence each other. This article represents our collaboration to understand these considerations in more detail, and especially to understand how they might lead to actual differences in the curriculum.

We draw on the authors' experience and knowledge as developers and instructors of the *Paradigms* (PJE, EG, and CAM) and the *Tutorials* (PJE, GP, and PSS) to articulate the different theoretical and practical considerations that shaped each curriculum. We also discuss example activities from each curriculum and explore how these activities exemplify the appropriate considerations. We use two curricula to frame this paper in order to draw from a diverse base of theoretical underpinnings and institutional constraints, and we use that base to propose broader conclusions about curriculum development. Author PJE has been a part of both teams and is in a unique position to be able to compare and contrast the *Paradigms* and the *Tutorials*.

We have chosen to draw example activities from the topic of quantum angular momentum because it forms a rich central point in undergraduate quantum mechanics. It is an advanced topic that builds on foundational concepts (like quantum measurements) and it also serves as a core element in analyzing three-dimensional quantum systems such as simple atoms. We hope this paper will thus serve as a useful addition to the published literature focused on the teaching and learning of angular momentum in quantum mechanics [4, 9, 10, 22, 40–43].

The main goal of this article is to describe the interactions between theoretical commitments about teaching and learning, practical/structural constraints, and the instructional activities that are developed in these contexts. Section II provides broad overviews of the two curricula. Then, we discuss each in detail: first *Paradigms* (in Sec. III) and then *Tutorials* (in Sec. IV). Within each section, we describe that curriculum's theoretical commitments about teaching and learning, the institutional structures in which each curriculum is embedded, and how both these theoretical and practical considerations impact the way activities are written and implemented. Sec. V discusses how the variety of theoretical and practical considerations inform each other, help developers make choices, and impact further curriculum development work at the upper-division level. We end in Sec. VI with a message for current and prospective quantum instructors.

II. BACKGROUND

Several obvious similarities between the curricula and their development stand out. In particular, the interplay of teaching and research serves as a strong foundation for the designers of each curriculum. Both the *Paradigms* and the *Tutorials* have been influenced by the research literature on both teaching and learning and on student understanding of various physics topics. Both curricula make substantial use of active engagement in the classroom, asking students to take ownership of their own thinking, and to interact with their peers and with instructors frequently.

The *Paradigms* and *Tutorials* classrooms also serve as research laboratories in which both students' ideas and instructional effectiveness have been studied. Although the two research groups have many differences, both emerge from a tradition of social constructivism [44] and share a practical research perspective that the research results should improve the teaching and learning of physics. Both groups also actively incorporate the findings of other research.

The research and development groups behind each curriculum operate using an iterative model whereby instructional materials are developed, implemented in the classroom, assessed, and then modified from year to year. Both groups view this iterative model as critical to curriculum development because it blends the results of formal research with practical pedagogical content knowledge of how students interact with particular physics topics and questions.

The substance of the *Paradigms* and the *Tutorials*, however, also demonstrate important differences, both in how they came to exist and in how they are implemented on a day-to-day basis. Below, we give a brief overview of the details of what each curriculum is and how it is enacted.

A. The *Paradigms in Physics* program

The *Paradigms in Physics* program is the set of core upper-division physics courses at Oregon State University (OSU). The centerpiece of the program is a sequence of junior-level courses (each of which is referred to as a *Paradigms* course). The content of the junior-year courses is structured so that each individual course focuses on a small number of key physical systems and relevant mathematical techniques [16, 34]. The courses are modular, meeting every day for seven hours each week for five weeks, including one week of integrated mathematical methods content. The classes are pedagogically interactive, making substantial use of a variety of student-centered techniques, including small whiteboard questions, small-group problem solving, computer visualization, integrated labs, and kinesthetic activities. These modular, junior-year courses are supplemented with a weekly (3-hour) computational lab and are followed in the senior year by a sequence of more conventional "capstone" courses that synthesize and extend the content from the junior-level courses. Since the beginning of the *Paradigms* program in 1996, the enrollment has increased from about 20 to about 45 students per year.

In this article, we focus on those *Paradigms* courses that include quantum mechanics content (and specifically, content relevant to angular momentum). McIntyre's textbook, *Quantum Mechanics: A Paradigms Approach* [45], was developed based on the first several years of the *Paradigms* program, and is now used as the textbook for all of the quantum-based courses. The first quantum *Paradigms* course, *Quantum Fundamentals*, uses a spins-first approach to introduce the postulates and fundamentals of quantum mechanics, providing students with a simple quantum system that is intended to serve as an analogy for more complicated, future quantum systems. As part of this course, students also begin to learn about position-space wave functions by studying the infinite square well potential.

In the quantum parts of *Central Forces* (offered toward the end of the junior year), students then explore increasingly more complicated quantum systems culminating in the hydrogen atom. Students learn to apply angular momentum concepts to each of these systems. Throughout the course, students are asked to identify similarities and new features compared to the spin and particle-in-a-box systems studied in *Quantum Fundamentals*.

In the senior-level *Quantum Capstone* course, students study advanced quantum systems both by combining previously studied simple systems (*e.g.,* spin-orbit coupling) and by learning and applying more advanced mathematical techniques.

The *Paradigms* began in 1996 as an experimental reimagining of the upper-division physics curriculum at OSU [16]. The design was led by OSU faculty members CAM, David McIntyre, and Janet Tate. Since then, the *Paradigms* has been continuously modified by a combination of the original faculty, new OSU faculty members, postdocs, and graduate students. These modifications have included the development of numerous activities and continuing efforts to resequence the junior-level physics content, including a recent major redesign, Paradigms 2.0.

B. *Tutorials in Physics: Quantum Mechanics*

Tutorials in Physics: Quantum Mechanics is a set of structured worksheets in the style of *Tutorials in Introductory Physics*, developed by the Physics Education Group at the University of Washington (UW). The worksheets are intended to supplement lecture instruction in undergraduate quantum mechanics by focusing on conceptual understanding and the building and application of key elements of the quantum model. The worksheets are divided into several sequences that each focus on some aspect of this model. One early sequence introduces students to Dirac notation, function spaces, changes of basis, and finding probabilities [10, 46]. Another early sequence focuses on quantum measurements and time dependence [32]. The sequence discussed in this article is a pair of tutorials on the topic of angular momentum in quantum mechanics [10].

The *Tutorials* are given in the junior-level quantum mechanics course at UW. The lecture portion of the course meets for 3 hours each week and has typically used the textbook by Griffiths [47], with most lectures carried out in the traditional format (*i.e.,* relatively little active engagement). Students also meet 1 hour each week in smaller recitation sections, in which the *Tutorials* are given. The total enrollment of the course has varied from about 50 students to more than 100 students.

Each tutorial includes activities administered after lecture instruction on the relevant topic. Students begin by completing a pretest (typically online) that gives them a first opportunity to express their ideas about the topic. Then, students attend a recitation section where they complete the tutorial worksheet in groups of 3-4, aided by graduate student teaching assistants. After the in-class worksheet, students are assigned 2-3 homework problems (in the same style as the in-class questions) intended to let students practice and extend the ideas considered on the worksheet.

The QM *Tutorials* were initially created during the early 2000s primarily by Andrew Crouse, Bradley Ambrose, and author PSS. They were developed at the request of faculty in the Department of Physics for use in the newly-created recitation sections for the upper-level quantum course [4]. Two major periods of development (alongside research on student understanding) followed: the first led by Crouse and PSS (2000-2007) and the second by PJE, GP, PSS, and Tong Wan (2011-2018).

III. QM PARADIGMS

In this section we articulate the instructional considerations (both theoretical and practical) of the *Paradigms*. We begin by describing the theoretical commitments that led to the initial and subsequent development of the *Paradigms* over the last twenty years. We then identify the practical structures that have also shaped the curriculum. We present the theoretical considerations before the practical ones because one of the defining features of the original design of the *Paradigms* was to eschew traditional course structures and requirements and mold new structures that fit with the developers' underlying philosophies. Finally, we describe example activities and how they enact the various instructional considerations.

A. Theoretical Commitments of the QM *Paradigms*

The *Paradigms* as a whole, including the QM *Paradigms*, were initially developed by a team of OSU physics faculty including many different individuals. The *Paradigms* program has continuously evolved since this initial development, an evolution that has resulted in both small and large changes to the curriculum. In this section, we aim to articulate the theoretical commitments that have most influenced the QM *Paradigms*. An initial list was drafted by author EG, and extensive discussions between authors PJE, EG, and CAM eventually led to the following five theoretical commitments:

P1 *Each individual must make their own set of cognitive connections* (Individual Cognitive Connections)

Since physics concepts, laws, and representations are strongly interconnected, students' knowledge structures about physics should also be strongly interconnected [48–50]. Students come to a course with a personal set of cognitive connections, though we do expect some similarities across students. Instructional activities should be rich enough so that different students can engage with the activities in different ways. Instruction should anticipate that different students will make different connections in any one activity. Therefore, sequences of activities should address a single topic from multiple perspectives or using multiple approaches so that students have many opportunities to make particular connections. A spiral structure should exist in the curriculum so that activities revisit topics or ideas in increasingly sophisticated ways. As stated by Manogue *et al.*, "Upper-division students must deal with problems of far greater complexity and must learn to see patterns which cross the boundaries of traditional physics subdivisions" [16].

P2 *Social interactions are instrumental to learning and doing physics* (Social Interactions)

Since physics ideas are a socially constructed description of the universe that changes over time, physics learners should learn to do physics by interacting with their instructors and other physics learners [51]. A major goal of upper-division physics instruction should be to bring students into the community of physicists and empower them to contribute to the construction of physical descriptions of the universe. At this level, instruction should help students begin to identify themselves as members of the physics community.

P3 *Instruction should respond to the ideas that the students in the room have* (Responsiveness)

Interactions should be a dialogue, with meaningful contributions from both students and instructors. In these interactions, students should participate in professional and productive discussions about physics. Classroom instruction should respond and adjust in real time to students' ideas [52]. To do this, instructors should learn about, respect, and value their students' ideas. Students should help by articulating their own ideas and by working to understand the ideas of their peers. Being wrong and refining ideas is a natural part of the process of constructing physics knowledge. Learning environments should facilitate interactions among instructors and learners and be made safe for learners to be wrong and refine their ideas.

Responsive instruction supports students in thinking about their own thinking [53]. Since professionals are metacognitively active, including planning and evaluating their solutions, students should also engage in these practices. Learning environments should go a step beyond demonstrating the instructor's thinking by providing explicit opportunities for students to make consequential choices when solving problems while the instructional team is present and able to provide support.

P4 *Physicists should be representationally fluent* (Representational Fluency)

Since physical systems and concepts may be externally represented in many ways, students need to become familiar with the set of representations used by physicists and be able to use these representations flexibly across physics contexts [54–57].

P5 *Students must learn how to ask appropriate questions about physical systems* (Epistemic Sophistication)

Learning involves asking and answering questions. It is important to know what kinds of questions are productive to ask. The kinds of questions that are meaningful are different for different subdisciplines of physics. Instruction should include explicit discussion of the kinds of questions that are productive for interrogating different physical systems in order to help students develop epistemic competence (*i.e.,* knowing about the nature of physics knowledge and learning physics) [58].

B. Practical Considerations of the QM *Paradigms*

We now identify several practical structures that grew out of the initial development of the *Paradigms*:

P6 *Class meets every day for 1 or 2 hour blocks for a total of 7 hours per week for five weeks* (Daily Schedule)

This schedule is demanding for both instructors and students, but an advantage is that students remember from one day to the next what they were doing. Activities can be long and can bleed over days. To accommodate this schedule, students take fewer courses at a time.

P7 *The course instructor leads activities* (Instructor as Leader)

The course instructor typically leads the activities and discussions. They can interrupt an activity with a short clarifying lecture and can easily adjust the sequencing of activities in response to student questions and discussion. Activities may introduce new content/topics; new vocabulary can be introduced immediately to name a concept that students have just "discovered" during an activity. Wrap-up discussions with the whole class provide an opportunity for synthesis discussions that are *not* nominally a part of the activity. These wrap-up discussions can happen immediately after an activity or the next day as a quick review.

P8 *Computers are available to students in class and in study rooms* (In-class Technology)

Computer visualization is incorporated into classroom activities. Each group of 2-3 students is provided a laptop computer (and some students bring their own devices). The instructor's computer can be displayed on monitors around the classroom for demonstrations.

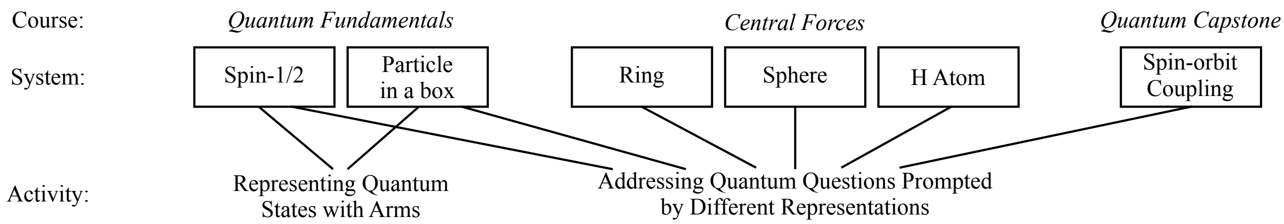

FIG. 1: The sequence of quantum systems considered across the QM *Paradigms*, with two example activities and where they occur and recur indicated.

Students learn (but may not be proficient with) *Mathematica*. A study room with computers running the same software is also available to students outside of class. In-class activities with computer visualization can easily be extended to homework.

P9 *The course instructor, graduate teaching assistant (TA), and undergraduate learning assistant (LA) are all present at every class meeting* (Multiple Instructors)

The enrollment is large enough that the course instructor cannot talk with each group during an activity—in-class TA/LA support is needed. Extensive pre-class discussions with the teaching team are a highly-valued opportunity to make sure everyone understands the goals and possible student conversations of the activities and to share observations about how the students are doing in order to make adjustments.

C. Example Activities that Exemplify the Instructional Considerations

Before discussing the examples in detail, we begin by situating the activities within the overall structure of quantum activities in the *Paradigms*, which are organized around a succession of physical systems that students explore in detail one after the other (each system is shown in a box in Fig. 1). In *Quantum Fundamentals*, students learn about systems with intrinsic angular momentum (spin-1/2 and spin-1), and are introduced to both the Dirac and matrix representations. At the end of *Quantum Fundamentals*, we introduce the particle-in-a-box system and the wave function representation. In *Central Forces*, we follow Goswami [59] by introducing three quantum systems: a particle confined to a ring, a particle confined to a sphere (the rigid rotor problem), and the (unperturbed) hydrogen atom. These three systems build on each other by introducing one new spatial dimension at a time to help students develop Individual Cognitive Connections (P1). The Ring introduces the z-component of angular momentum and the concept of degeneracy. The Sphere introduces L^2 and the other components of angular momentum. The H Atom introduces all three quantum numbers n, ℓ, and m. Lastly, the *Quantum Capstone* (a senior-level course) uses the basic quantum building blocks from the junior year to look at quantum systems that are an elaboration on earlier ones (*e.g.,* non-degenerate perturbation theory, degenerate perturbation theory, spin-orbit coupling, addition of angular momenta, *etc.*)

Within each of the quantum mechanical systems described above, students participate in a variety of activities, such as solving for eigenstates, exploring the features of different representations, and determining possible measurement outcomes and probabilities. Below, we describe two foundational activities: a kinesthetic activity aimed at representing spin-1/2 quantum states and a small-group activity focused on multiple representations.

1. *Representing Quantum States with Arms*

The QM *Paradigms* begin with a spins-first approach [45] where students use a computer simulation (P8) of Stern-Gerlach experiments [18] to explore the postulates of quantum mechanics and develop intuitions about quantum measurements. The students learn that the distribution of outcomes for identically prepared particles determines a quantum state (Fig. 2a). Students use the results of Stern-Gerlach experiments to determine Dirac and matrix representations of the states of spin-1/2 particles [45, p.17-25]. During these calculations, students are introduced to the fact that the relative phase between terms determines the state; multiplying a state by an overall phase does not change the state.

After using diagrams of experiments, histograms of probabilities, matrices, and Dirac notation to represent quantum states (Fig. 2a-d), students do a kinesthetic activity [60–63] where they work in pairs to represent spin-1/2 states with their arms. The students stand shoulder to shoulder and use their left arms to sweep out a complex plane: the real axis is forward, parallel to the ground and the the imaginary axis points vertically upward (Fig. 2e). The students use their left arms so that, when looking at their own arms, the students see the complex plane in the standard orientation. This activity occurs about 5 instructional hours after the students have done a similar activity where each student represents a single complex number with their arm.

The student standing on the left in each pair is told they should represent the probability amplitude (complex coefficient) of the spin-up-in-the-z-direction component of the spin

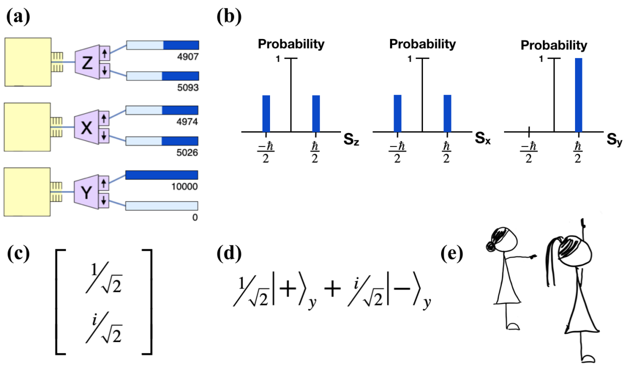

FIG. 2: Various representations of a spin-1/2 state: (a) schematic of the results of Stern-Gerlach experiment, (b) histograms of probabilities of values of spin, (c) matrix notation, (d) Dirac (bra-ket) notation, (d) "arms" notation

state, and the person on the right represents that of the spin-down-in-the-z-direction component. The instructor writes a state on the board, either in matrix or Dirac notation and says:

> *Instructor: Show me this state.*

For example, if the state were $|+\rangle_y$, the students could arrange themselves so that the student standing on the left points forward with their arm parallel to the ground and the right students points vertically upward, as in Fig. 2e (other arrangements that preserve the relative angle between the students' arms are also correct).

After the students have represented a few states, the instructor then asks:

> *Instructor: How can you tell the difference between $|+\rangle_y$ and $|-\rangle_y$?*

The class then discusses that for $|-\rangle_y$, if the student on the left is pointing forward, parallel to the ground, the student on the right should point vertically downward, meaning that it is the angle between the two arms that determines the state.

The instructor then asks:

> *Instructor: Show me $e^{i\pi}|+\rangle_y$?*

The students could arrange themselves so that the left student points backward, parallel to the ground and the right student points vertically downward. The class then discusses whether or not this state is equivalent to $|-\rangle_y$ (it is not—it has a different relative phase!).

In this activity, students translate either matrix or Dirac representation (ideally, both) for a spin-1/2 system into "arms" notation, supporting the development of Representational Fluency (P4). Although not widely used by physicists (we invented it), "arms" is a pedagogically useful representation [60, 64]. Similar activities with arms occur later in the course to represent time dependence and then spin precession. The students collaborate in pairs to create the arms representations, and students can compare themselves to other pairs' configurations in the room (Social Interactions—P2). The instructor can see each pair of students and can therefore point out variations and adjust which states the students are asked to represent to accommodate the level of understanding in the room (Responsiveness—P3). The prompts are fundamentally open-ended, and the fact that quantum states have an arbitrary overall phase means that different pairs of students can make different correct choices and the class can acknowledge these different choices (Individual Cognitive Connections—P5).

2. *Addressing Quantum Questions Prompted by Different Representations*

We now describe a touchstone activity sequence entitled: "Angular Momentum and Energy for a Particle on a Ring" [65]. The sequence occurs at the beginning of *Central Forces*, immediately after lecture content on finding the energy eigenstates for a particle confined to a ring, $\Phi_m(\phi) = \sqrt{\frac{1}{2\pi r_0}} e^{im\phi}$.

In the sequence, students are given the first two quantum states and asked questions 1-4 in Fig. 3. Students are given the two states one at a time in quick succession (the other two states can be given either in class or on homework as a separate activity). The four states are in fact equivalent but are represented successively in Dirac, matrix, wave function (individuated), and wave function (compact) notations. While the questions for each state are the same, the techniques for answering them differ based on the representation used. Thus, the sequence attends to Representational Fluency (P4) by asking exactly the same set of questions for the same quantum state but prompted by different representations.

As with most *Paradigms* activities, the students work together in 3-person groups. Social Interaction (P2) is promoted by having groups sit at tables with movable chairs around a shared whiteboard, and every student has a marker and can write on this shared brainstorming space.

The reader is encouraged to try each version personally. The Dirac and matrix versions are the most straightforward since the probability amplitude in each case is just the coefficient of each eigenstate. The fact that one needs to add the probabilities (squared norms of the probability amplitudes) in the case of states with degenerate energies is novel and a precursor to later questions that ask the probability of finding the particle in a particular region of space. For the case of individuated wave functions, the individual eigenstates are still readily identifiable, but the probability density and the normalization of the eigenstate have been blended in a way that students must sort out. The compact wave function notation is trickiest, and most students stumble at some point in the calculation. This question can best be posed in homework where students have time to work out the necessary analogue of Fourier Series on their own (P1).

Considerable attention is spent throughout the QM

Quantum Calculations on a Ring I-IV

In this activity, your group will carry out calculations on each of the following normalized abstract quantum states on a ring:

Dirac notation	$\lvert\Phi_a\rangle = \sqrt{\frac{1}{20}}\lvert 3\rangle + \sqrt{\frac{9}{20}}\lvert 1\rangle + \sqrt{\frac{9}{20}}\lvert -1\rangle + \sqrt{\frac{1}{20}}\lvert -3\rangle$
Matrix notation	$\lvert\Phi_b\rangle \doteq \begin{pmatrix} \vdots \\ 0 \\ \sqrt{\frac{1}{20}} \\ 0 \\ \sqrt{\frac{9}{20}} \\ 0 \\ \sqrt{\frac{9}{20}} \\ 0 \\ \sqrt{\frac{1}{20}} \\ 0 \\ \vdots \end{pmatrix}$
Wave function notation (individuated)	$\Phi_c(\phi) = \sqrt{\frac{1}{40\pi r_0}}\left(e^{i3\phi} + 3e^{i\phi} + 3e^{-i\phi} + e^{-i3\phi}\right)$
Wave function notation (compact)	$\Phi_d(\phi) = \sqrt{\frac{8}{3\pi r_0}}\cos^3\phi$

For each of the following questions, state the postulate of quantum mechanics you use to complete the calculation and show explicitly how you use that postulate to answer the question.

1. For each state above, what is the probability that you would measure the z-component of angular momentum to be $-3\hbar$? $0\hbar$? $2\hbar$?
2. What other possible values for the z-component of angular momentum could you have obtained with non-zero probability?
3. For each state, what is the probability that you would measure the energy to be 0? $\frac{4\hbar^2}{2I}$? $\frac{9\hbar^2}{2I}$?
4. If you measured the energy, what possible values could you have obtained with non-zero probability?
5. How are the calculations you made for the different state representations similar and different from each other? Be prepared to compare and contrast the calculations you made for each of the different representations (ket, matrix, wavefunction).

FIG. 3: The *Paradigms* activity "Angular Momentum and Energy for a Particle on a Ring." The activity involves four different representations of the given quantum state (above) and asks the same set of questions for each state (below). (Note that $I = Mr_0^2$ is the moment of inertia for the particle on the ring.)

Paradigms to helping students develop Epistemic Sophistication (P5) by repeatedly asking the same questions with the same wording. This sophistication is important because it is only possible to ask a few kinds of questions about simple QM systems, and because these questions are quite different in nature from the questions that can be asked about *classical* systems. For example, most classical mechanics questions are about the position, velocity, or acceleration of a particle, whereas most quantum mechanics questions are about the possible outcomes of a measurement and the corresponding probabilities.

Typically only the first two states are considered in class, with the others assigned as homework. The instructor can decide on the fly (P7) which states to consider in class depending on how much help the students need understanding the nuances of the different representations. This feature of the sequence thus demonstrates an important intersection between two of the theoretical commitments: the Responsiveness (P3) of the instructor to the ideas in the classroom and the different Individual Cognitive Connections (P1) that students might make both in class and on the homework.

A whole-class discussion addresses the crucial question 5 (see Fig. 3), which further helps students consolidate Representational Fluency (P4). The whole-class discussion also permits further Responsiveness (P3) by allowing the instructor to tailor the exact nature of the discussion to the ideas that the instructional team observed while helping students. Often, student groups may be asked to present their results so that both typical and unexpected solutions are brought forward and discussed.

It is essential to position learning opportunities appropriately to help students make Individual Cognitive Connections (P1). For example, the Ring is similar to the infinite square well potential, which students have studied in *Quantum Fundamentals*, so the students are not overburdened by extensive new content. However, the Ring has periodic, rather than fixed, boundary conditions, which means that the energy eigenvalues are degenerate. This is the first QM system students encounter that has both a wave function representation and degeneracy. Finally, the Ring is a one-dimensional QM system, so the complications posed by more dimensions and more quantum numbers is not present.

This activity sequence mirrors several other activities given throughout the QM *Paradigms*. In the preceding *Quantum Fundamentals* course, students consider very similar activities where they carry out parallel calculations in Dirac and matrix notation for a spin-1/2 system, and in Dirac, matrix, and wave function notation for the particle-in-a-box system. In *Central Forces*, students will later do the same in the context of two progressively more complicated systems: first for a particle confined to a sphere and second for the hydrogen atom. Lastly, in the *Quantum Capstone* students consider a system with both spin and orbital angular momentum. The cyclical nature of the activities means that students have additional opportunities to make different Individual Cognitive Connections (P1) in the subsequent activities.

IV. QM TUTORIALS

We now discuss *Tutorials in Physics: Quantum Mechanics* in the same fashion as we discussed the *Paradigms in Physics* above. We begin by describing the practical structures that led to the fundamental format of the QM *Tutorials*. Then, we identify the theoretical commitments that were most important to the efforts to develop specific tutorial activities. Lastly, we highlight a specific sequence of tutorial exercises (for the topic of quantum angular momentum), and detail how the development of those exercises was influenced by the instructional considerations (both theoretical and practical).

A. Practical Structures of the QM *Tutorials*

The QM *Tutorials* were developed over a period of about fifteen years at UW. As described in Sec. II B, the overall style was originally modeled on the introductory *Tutorials*, which had already been in use for many years. (The introductory *Tutorials*, in turn, were heavily influenced by the *Physics by Inquiry* curriculum used primarily to prepare future science teachers [66].) Around the time the QM *Tutorials* were developed, the upper-division quantum mechanics courses at UW added a fourth credit-hour in the form of a recitation, and some faculty in the department expressed a desire to have materials similar to the introductory *Tutorials* used in these sections. Part of the reason for the similarity between the two types of tutorials is that some of the practical structures that led to the development of the introductory *Tutorials* (*e.g.,* use in recitation sections as a supplement to large lecture sections) were also true of the quantum mechanics course at UW. Because the QM *Tutorials* were very strongly shaped by these constraints, we present them first, followed by the theoretical commitments in the following section. For ease of reference, we begin the numbering of the considerations from 10.

T10 *The QM Tutorials are supplementary to lecture instruction* (Supplementary Curriculum)

The *Tutorials* are a supplementary curriculum, in that they rarely introduce new content. Therefore, the role of the tutorials is to bolster and expand students' understanding of topics introduced in lecture.

T11 *The QM Tutorials are aimed at small-group "recitation" sections* (Recitation Sections)

The *Tutorials* are designed for and administered in "recitation sections" with 20-25 students, rather than in a lecture hall with 100-200 students. In a tutorial exercise, "the instructor is expected to act more like a facilitator of discussion than a dispenser of knowledge" [67]. Furthermore, the *Tutorials* are typically taught by teaching assistants (TAs), who may have varying levels of experience with either the physics material or with implementing active engagement in the classroom. It is therefore essential to provide appropriate preparation for the TAs in the *Tutorials'* style of instruction, though additional structure is included to accommodate instructors with varying levels of experience and preparation. The reliance on TAs requires that activities be sufficiently scaffolded and each question prompt is precisely worded to limit the effect of different tutorial instructors.

T12 *The QM Tutorials use limited technology* (Limited Technology)

The *Tutorials* are given in low-technology rooms without access to computer simulations. The tasks are given to the students on paper and designed to be completed on a large working space such as a tabletop whiteboard, and then copied onto each student's worksheet.

B. Theoretical Commitments of the QM *Tutorials*

The QM *Tutorials* were born out of the practical structures listed above. The goal during development was to make as big a difference in student understanding as possible in only 50 minutes a week. The *Tutorials* include contributions from a variety of researchers at UW, each bringing different perspectives to the development of the curriculum. Below, we summarize the theoretical commitments that authors PJE and GP identified as most influential to the activities they developed at UW. The commitments were identified by reflecting on the curriculum, with additional insights from prior publications regarding *Tutorials in Introductory Physics*. After extensive discussion between authors PJE and GP, as well as discussions with Paula Heron at the University of Washington, we articulate the following five theoretical commitments that most influenced our development of the QM *Tutorials*:

T13 *Having a coherent framework is helpful for reasoning in both familiar and new contexts* (Coherent Framework)

We believe that having a coherent framework for physical concept(s) helps promote successful reasoning, especially when transferring knowledge from one context to another. By framework, we mean a set of physics rules and principles coupled with the criteria for when they apply and the knowledge of how to use them. While the *Tutorials* use a variety of individual strategies to improve student understanding (many targeting particular difficulties, as described below), the "broader structure of experiments and exercises [is] intended to guide the construction of a *coherent conceptual framework*" [68]. Particularly common mechanisms for helping students construct a coherent framework are to focus student attention on fundamental concepts, encourage the creation of links between concepts, and promote the use of multiple representations.

T14 *Many student responses are predictable and transcend context* (Predictable Responses)

We recognize that students bring a broad array of knowledge into the classroom, both from prior courses and from everyday experiences. We consider this knowledge to be relevant to instruction. Research has revealed the existence of certain patterns of answers or chains of reasoning that are prevalent [69, 70]. When these patterns lead to incorrect answers, they are given the term *difficulties*. Heron notes that a "difficulty is not the specific idea or reasoning pattern, it is the *use*, or *misuse*, thereof" [68]. Many tutorials are specifically designed to address such difficulties that have been identified through research.

T15 *Understanding is more than just (symbolic) answer-making* (Non-symbolic Reasoning)

We view deep understanding of physics as being marked not solely by an ability to give correct answers, but also by the ability to explain *how* an answer is determined (reasoning) and to interpret the *meaning* of an answer (sensemaking) [71, 72]. As articulated by Shaffer and McDermott, the goal of a tutorial "is not to deliver additional information but to help students deepen their conceptual understanding and develop skill in scientific reasoning" [67]. Since lecture instruction is often mostly symbolic, the *Tutorials* frequently ask students to provide or interpret both verbal explanations and graphical representations in their explanations.

T16 *New knowledge is constructed on existing knowledge in a social environment* (Social Constructivism)

Driver *et al.* argue that scientific knowledge is "socially negotiated" and that education should acknowledge this fact [44]. In particular, the developers of tutorials, as Heron notes, "assume that learners construct new knowledge on the basis of their existing knowledge .... Prior knowledge is viewed both as the foundation upon which new knowledge is built, as well as the building material" [68]. It is especially important that this knowledge is built *by the students*, rather than conveyed by an instructor. Furthermore, we recognize that knowledge is typically built in a social environment between multiple learners and instructors, and that communication and interaction are therefore crucial.

T17 *Structured inquiry facilitates learning* (Structured Inquiry)

The preface to *Tutorials in Introductory Physics* claims that "it can be difficult for students who are studying physics for the first time to recognize what they do and do not understand and to learn to ask themselves the types of questions necessary to come to a functional understanding of the material" [37]. Student learning, therefore, benefits from *structure* that helps students learn to ask *themselves* the right questions at the right instants in time. By asking these questions, students are then able to gain individual skills along with the understanding of when and how to apply those skills. While structure plays an important role, sufficient room is simultaneously given for students to express their ideas and explore the concepts.

C. Example Activity that Exemplifies the Instructional Considerations

We now present a sequence of exercises from a single tutorial and discuss how they are influenced by the theoretical commitments and practical structures outlined above. We primarily restrict ourselves to discussing this single sequence of exercises, but where necessary we also discuss closely related exercises that precede or follow the chosen example. We emphasize that this example only demonstrates one instance of how the commitments have impacted the curriculum, and that the same commitments have resulted in different decisions about the structure of other tutorial activities.

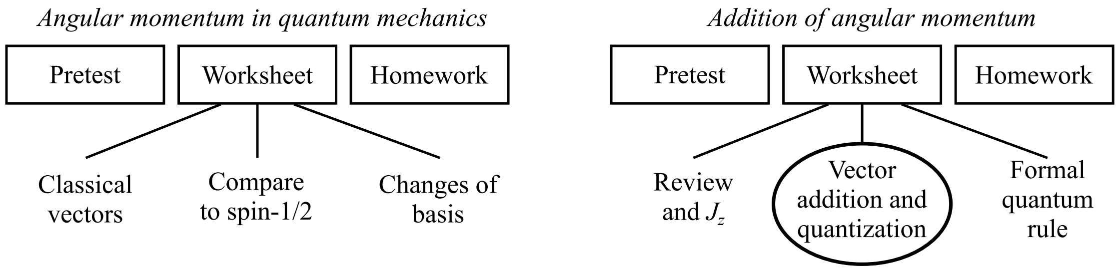

FIG. 4: The sequence of activities associated with the angular momentum tutorials. Each tutorial consists of a pretest, worksheet, and homework assignment, all given after lecture instruction on the corresponding topic. Each worksheet is further divided into three activities—in this article, we describe the Vector addition and quantization activity in detail.

As discussed in Sec. II B, the angular momentum sequence is composed of two tutorials: *Angular momentum in quantum mechanics* and *Addition of angular momentum* (see Fig. 4). The example that we introduce is taken from the middle of the second tutorial in the sequence (circled in Fig. 4). Prior to working on this tutorial, students have had lecture instruction on angular momentum, the hydrogen atom, spin-1/2, and addition of angular momentum. The students have also completed the first tutorial in the sequence, along with several previous tutorials focusing on inner products, time dependence, and measurements.

The primary learning objective for this activity is that students should be able to determine the possible outcomes of a measurement of J^2 (the square of the total angular momentum, $\vec{J} = \vec{L} + \vec{S}$) for a quantum state written in terms of the quantum numbers l, m_l, s, and m_s (this is sometimes known as the *uncoupled* basis). The general answer is that the allowed values of j (the quantum number associated with J^2) range from $|l - s|$ to $l + s$ in integer steps. This answer can be counterintuitive, and students frequently believe that $j = l + s$ is the only possible value [10]. The possible outcomes of a measurement of J^2 are then given by $j(j+1)\hbar^2$ for each possible value of j. The activity described below leverages students' understanding of vector addition in *classical* physics contexts to build an intuition for why there are multiple possible answers for j and to determine what those answers might be.

The overall structure of the activity uses the *elicit-confront-resolve* strategy that has been effective in other instructional contexts [70]. Earlier in the tutorial (and on the pretest), students predict whether or not the magnitude of $\vec{J}$ will be well-defined (that is, whether or not it has only one possible value). Most students get this prediction wrong, and they tend to pull from a diverse set of resources when answering this and other questions about angular momentum measurements [10]. Since students seem to lack a Coherent Framework (T13) for quantum angular momentum, the *confront* stage of the activity asks students to construct their own answer based on what they know of classical vector addition and quantization. Afterwards, they reflect on their original prediction (along with alternate possible predictions).

Below, we discuss the activity itself in two parts: (1) using classical knowledge to build quantum understanding and (2) reflecting on possible explanations and resolving inconsistencies. In each part, we describe the exercises given to the students, followed by a discussion of how the exercises highlight the theoretical and/or practical considerations introduced in the Sec. IV B. We then discuss two follow-up exercises that reinforce some aspects of the chosen activity.

1. Using classical knowledge to build quantum understanding

Throughout the *Addition of angular momentum* tutorial, students consider an electron in the state $|l, m_l; s, m_s\rangle = |2, 0; 1/2, 1/2\rangle$. Students are first asked to recall relevant knowledge about the angular momentum operators L and S for this state, *e.g.,*

> Determine the magnitude of the orbital angular momentum vector, $\vec{L}$, for this particle. Approximate this value to two decimals in units of $\hbar$. (*Hint:* It is *not* just $l\hbar$.)

This question is immediately followed by the same question for $\vec{S}$, and students are then asked whether or not the directions of $\vec{L}$ and $\vec{S}$ are well defined.

The core of the activity asks students to use the known quantum values for the magnitudes of L and S (approximately 2.45$\hbar$ and 0.87$\hbar$, respectively), and the lack of certainty about their directions, to construct the *classically* possible magnitudes of the total angular momentum ($\vec{J} = \vec{L} + \vec{S}$):

> For this sequence of questions, suppose that angular momentum were *classical* (*i.e.,* that the allowed values for angular momentum were continuous rather than discrete).
>
> 1. What is the largest possible value for the magnitude of the total angular momentum vector, $\vec{J}$, for this particle? What is the smallest possible value?

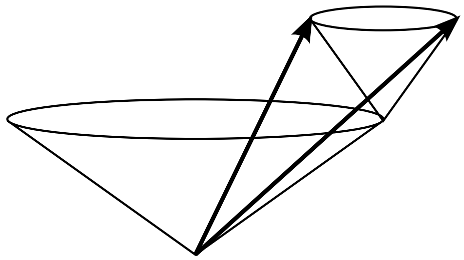

FIG. 5: A graphical representation of the inherent uncertainty in quantum angular momentum. The large cone represents L, the small cone S, and the two arrows represent two possible (classical) angular momentum vectors with *different* magnitudes.

2. Draw alignments of the vectors $\vec{L}$ and $\vec{S}$ corresponding to at least three different values for the magnitude of $\vec{J}$.

3. Determine both the largest and the smallest possible values of J^2 for this particle, assuming that angular momentum can be treated classically. Approximate these values to two decimals in units of $\hbar$.

The development of this exercise was strongly influenced by Social Constructivism (T16). In particular, each group of students *constructs* the classical behavior of the sum of two vectors whose relative directions are unknown. In early drafts of the tutorial, students were asked to use a graphical version of this argument using cones (see Fig. 5), which is presented in some textbooks [47]. We found that this representation often proved too difficult and misleading for students' first reasoning with a classical argument, and this exercise was moved to the tutorial homework (see Sec. IV C 3). In this instance, the Limited Technology (T12) available in the classroom prevented us from using a computer simulation to help students with this visualization, and so we instead chose to develop a task students could complete by hand.

In the next exercise, the students return to the quantum system and make a list of the first four allowed (half-integer) values of j and the corresponding eigenvalues $j(j+1)\hbar^2$. They then compare this list with the classical range they determined previously to build a reasonable set of quantum values for J^2 for the given system, culminating in the following question:

> What are the possible values of j for this electron? What are the corresponding values of J^2? Explain.

Here, the students are finding an answer for themselves that can be used both to assess their predictions and to account for the fact that the quantum rule gives multiple possible values.

2. *Reflecting on possible explanations and resolving inconsistencies*

The activity concludes with questions to help students reflect on their answers. The first is a student dialogue:

> Consider the discussion between two students below.
>
> Student 1: *"Originally, we knew $l = 2$ and $s = 1/2$. Since $J = L + S$, we just add l and s to get j, which would be equal to $5/2$ for this particle."*
>
> Student 2: *"You can't do that because J, L, and S are vectors. Since L and S could point in any direction, the magnitude of J could be any number between the magnitude of $L-S$ and the magnitude of $L+S$, which for this particle would be $1.58\hbar < J < 3.32\hbar$."*
>
> Both students are incorrect. Identify the flaws in each student's reasoning. Explain.

The use of a student dialogue with common incorrect answers is a strategy used throughout the *Tutorials*. This strategy was primarily chosen to address students' Predictable Responses (T14). In particular, Student 1's statement highlights the incorrect line of reasoning that our research found was most common in response to questions about J^2 [10]. Student 2's statement also corresponds to reasoning that is commonly given by students, and is included here to help keep students from overgeneralizing the classical portion of the activity. This dialogue specifies that both statements are incorrect, while in other dialogues, students are asked to agree with one or more of the statements. Because this question specifically asks students to identify how each line of reasoning is incorrect, students must go *beyond* just providing an answer and instead explore the reasoning *underlying* the answer. This intersection between the *Tutorials'* recognition of Predictable Responses (T14) and the value of Non-symbolic Reasoning (T15) often leads to particularly powerful learning opportunities for students

Student dialogues are included in tutorial activities very frequently—in fact, there are very few worksheets that do not include at least one student statement or student dialogue. The dialogues exemplify an *intersection* between the theoretical and practical considerations of the tutorials. In addition to the theoretical commitments mentioned above, student dialogues are a very clear example of Structured Inquiry (T17). Practically, the student statements encode pedagogical content knowledge into the text of the activity itself, so that it is easy for TAs in Recitation Sections (T11) to reference even if they do not have the relevant prior classroom experience.

The last question in the activity asks students to resolve any inconsistencies between their answers and an earlier exercise in which the students are asked to "predict whether or not the magnitude of the total angular momentum, $\vec{J}$, for this electron will be well-defined." This question serves as the final step in the *elicit-confront-resolve* strategy used to structure the overall activity. Explicitly asking students to resolve any inconsistencies is a crucial aspect of Structured Inquiry (T17), as students will often proceed without resolving, or sometimes without even *noticing*, an inconsistency.

At the end of the activity, students are asked to check their answers with an instructor before proceeding (this is very common in the *Tutorials*). The role of the check-in is for students to repeat their explanations verbally and for the instructor to ask probing follow-up questions to get a sense for both their understanding of the Coherent Framework (T13) and the sophistication of their Non-symbolic Reasoning (T15).

3. *Understanding in alternate representations*

After the conclusion of the activity above, students work on a third section in which they are reminded of the quantum rule for determining the allowed values of j (covered in lecture prior to the tutorial) and asked to verify, extend, and formalize their findings from the prior section. In their tutorial homework, the students are asked to consider a common textbook representation for angular momentum (the "cone" representation—see Fig. 5) and to describe how this representation might help explain the fact that there is more than one possible value for J^2. The homework also questions students about the limitations of the cone representation for describing a quantum system (*i.e.,* the angular momentum for a quantum object cannot be represented by a single, well-defined vector).

Both of these follow-up activities ask students to make connections to bolster their understanding of a Coherent Framework (T13) for quantum mechanics. Students are asked to revisit the symbolic rule for the allowed values of j in order to link the Non-symbolic Reasoning (T15) from the tutorial with the symbolic answer introduced in class. This is especially important in this case because so many students fail to use this rule when making predictions at the beginning of the tutorial, despite the fact that the rule has been previously covered in lecture. Returning to the symbolic rule after an alternate conceptual understanding has been developed is intended to help cement students' ability to use the rule productively in future reasoning. Similarly, considering the same classical argument as in the tutorial using a different *representation* (the cones in Fig. 5) helps students practice the non-symbolic reasoning in a new way so that students practice *using* the reasoning and not just using the rule.

V. DISCUSSION

The previous two sections described instructional considerations of two comprehensive quantum mechanics curricula: the *Paradigms in Physics* and *Tutorials in Physics: Quantum Mechanics*. We explored both the theoretical commitments of each curriculum as well as the practical structures within which each curriculum is administered. We then identified the *impact* of these beliefs and structures on the curricula themselves using example activities to highlight the canonical choices of each set of developers. We now discuss what we have learned from examining the *Paradigms* and the *Tutorials together*.

A. Interplay between Theoretical Commitments and Practical Structures

Both the *Paradigms* and the *Tutorials* were influenced by the institutional environment in which they were developed. Despite making distinctions between theoretical commitments and practical structures in the previous two sections, we recognize that they can inform each other and a clean distinction between them is somewhat artificial. Additionally, each curriculum has a different relationship to these considerations: in the *Paradigms*, the theoretical commitments drove changes to the practical structures, whereas in the *Tutorials*, the practical structures informed the theoretical commitments that were adopted.

The *Paradigms* were a purposeful redesign of the middle- and upper-level physics curriculum intended to center the theoretical commitments, commitments that *dictated* the practical structures, especially the Daily Schedule (P6), In-class Technology (P8), and Multiple Instructors (P9). The designers of the *Paradigms* were so committed to the theoretical commitments that we were willing to go to considerable trouble to *change* the practical structures: establishing consensus amongst the entire faculty for change; working with the registrar's office to implement a different weekly schedule and course length; and remodeling a classroom for interactive engagement and computer use.

The *Tutorials*, on the other hand, were designed to be a Supplementary Curriculum (T10) and given in Recitation Sections (T11). These constraints were inherited partly from the introductory *Tutorials*, which were themselves a compromise to bring strategies and methods that had been successful in *Physics by Inquiry* into the broader undergraduate physics curriculum. But they also arose from departmental circumstances substantially different from those surrounding the development of the *Paradigms*—namely, that the number of physics majors at UW was and continues to be very large and that there was no department-level effort to redesign course sequences. Rather, efforts were dedicated to improving student understanding by supplementing lecture instruction in recitation sections using research-based materials. For these reasons, the theoretical commitments of the *Tutorials* are strongly influenced by what can be achieved in the more constrained environment of a weekly 50-minute recitation section.

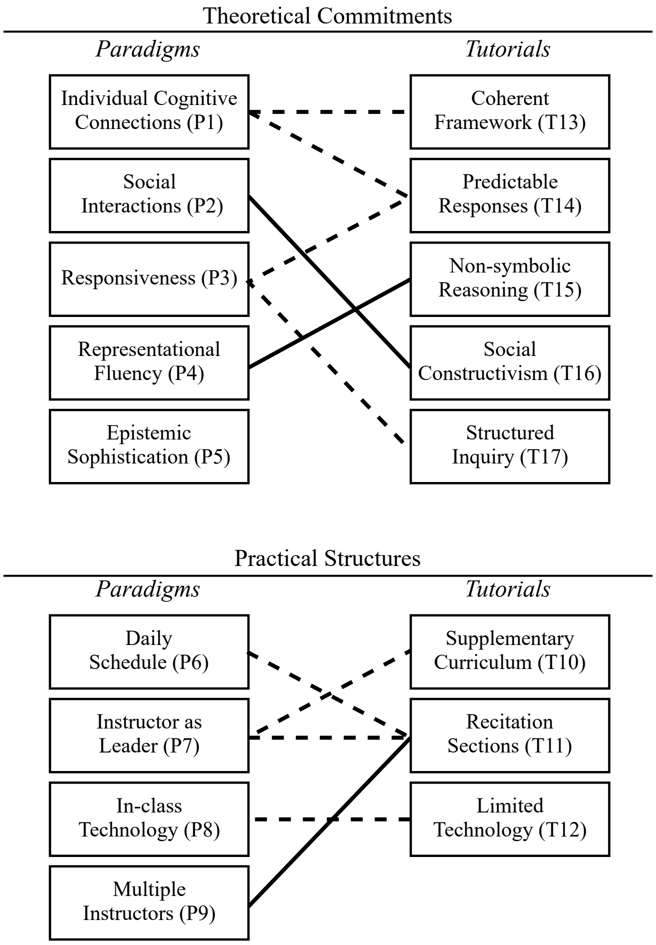

FIG. 6: Instructional considerations for the *Paradigms* and the *Tutorials*. The theoretical commitments are listed first, followed by the practical structures. Considerations that we identified as in alignment are connected by solid lines, while considerations that are in tension are connected by dashed lines.

B. The Different Curricula Prioritize Different Theoretical Commitments

In reflecting on the two sets of instructional considerations, we observe both similarities and differences (see Fig. 6). We choose to focus primarily on the theoretical commitments. Two unsurprising similarities stand out: both curricula value social constructivism (P2 and T16) and representational fluency (P4 and T15). Social constructivism [44] is a theoretical background that has influenced the research groups at both OSU and UW (and many others), and underlies much of the research literature on interactive engagement. Representational fluency is the idea that the ability to understand different representations and to be able to go fluidly back and forth between them is helpful in physics contexts. Among the remaining theoretical commitments, we articulate three differences in priority between the two curricula.

First, the *Tutorials* are built primarily to target Predictable Responses (T14), where the *Paradigms* prioritize Responsiveness (P3) to attend to students' ideas and to promote Individual Cognitive Connections (P1). In other words, the *Tutorials* are more structured in an effort to help all students with one or two particularly prevalent difficulties, while the *Paradigms* are more agile in an effort to help each student with more individualized concepts.

Each curriculum, however, also acknowledges the theoretical commitment that the other prioritizes. That is, the developers and instructors of the *Paradigms* are aware of the most common student ideas, and are well prepared to deal with them when they arise. Conversations that have previously helped students come to new understandings in the classroom are well documented in the curricular materials. Similarly, the developers and instructors of the *Tutorials* know that some students are likely to raise issues that the worksheets are not intended to address. TAs are trained to use Socratic questioning so students can articulate their reasoning and the TA can then respond to each student's needs.

The difference described above leads into a tension between the Responsiveness (P5) of the *Paradigms* and the Structured Inquiry of (T17) of the *Tutorials*. The open-ended nature of the prompts in the *Paradigms* promotes metacognition by forcing the students to monitor *their own* reasoning. This metacognition is supported by the responsiveness of both the instructor-student interactions and the whole-class discussions [53]. For example, students are frequently asked to share their (diverse) solutions with the class as a whole, allowing each student to explore a larger set of experiences than is possible for a single group alone. The *Tutorials* instead often use a formal structure in which students are asked to invoke particular knowledge elements in a systematic way intended to help them follow certain productive chains of reasoning in contexts that grow more and more difficult [70]. This kind of structure is followed in almost all of the tutorials, with the long-term goal that students will eventually learn how to ask their own important questions.

A last distinction between the theoretical commitments of the two curricula arises from the *Paradigms'* commitment to each student building on their own prior knowledge in an order that makes sense to them (P1). Multiple opportunities are provided for students to pick up on connections they may have missed. For example, different students might make different connections while working on the Ring activity described in Sec. III, but the additional connections they make in the subsequent Sphere and H Atom activities further each student's knowledge toward a more sophisticated network of ideas. The *Tutorials* instead aim to have each student build certain knowledge elements and connections at the same point in time, in order to build a Coherent Framework (T13), so that those elements can be used for the next activity in the sequence.

The tensions identified above highlight clear differences in the priorities of the designers of the two curricula. However, in each case the *Paradigms* commitment and the *Tutorials* commitment are statements about parallel aspects of the same underlying principle. Both curricula, for example, attend to ideas that students have, but the *Paradigms* attend more directly to ideas definitely present in the classroom while the *Tutorials* focus more strongly on ideas that research has shown to be especially common. Looking at the full set of commitments for both curricula, all the authors find ourselves in a position where we do not disagree (for the most part) with each other's theoretical commitments, even though we *prioritize* differently.

C. Accounting for Differences in the Curricula with Theoretical and Practical Considerations

Several obvious differences between the activities in the *Paradigms* and the *Tutorials* emerge when we examine the activities discussed in Sections III and IV. We attempt to account for how these differences arose given the various theoretical and practical considerations.

One of the most obvious differences is the style of prompt, which arises primarily out of the tensions between the commitment of the *Paradigms* to Responsiveness (P3) and to students' Individual Cognitive Connections (P1) and the Structured Inquiry (T17) that the *Tutorials* use to focus on students' Predictable Responses (T14). The *Paradigms* tend to use a small number of short, open-ended prompts for any single activity. While these prompts may occasionally be written on handouts, they are often written or delivered verbally by the instructor one at a time. The open-ended nature of the prompts give students room to recruit a broad set of prior knowledge and explore and regulate new connections. This prompt structure allows the instructor to respond to students ideas by taking up students' language and changing subsequent questions as warranted. As a result, examples and problems that have been informed by the results of formal research are improved over time by classroom dialogue.

The *Tutorials*, on the other hand, are composed of worksheets that use a Structured Inquiry (T17) approach that guides students to consider particular predetermined lines of reasoning. These lines of reasoning are almost always Predictable Responses (T14) that are the result of in-depth research into student ideas about a given topic. The decision to structure the inquiry in this focused way, instead of using a more open-ended form of inquiry, is in large part due to the practical structures of the *Tutorials* as a Supplementary Curriculum (T10) given in Recitation Sections (T11) that have limited time.

Another difference between the curricula is the specific content (and the amount of content) covered. In this paper, we focus on the topic of angular momentum in quantum mechanics, but we suspect that similar differences are present in other quantum contexts. As a complete curriculum, the QM *Paradigms* must first introduce and then expand upon angular momentum in quantum mechanics. The overarching structure is spiral, in which students explore angular momentum several times (initially as spin, then as orbital angular momentum through the cycle of Ring, Sphere, and H Atom, and finally combining orbital angular momentum and spin), with each successive instance adding some complexity to the topic while also revisiting the fundamentals introduced previously. In contrast, the *Tutorials* do not introduce angular momentum, but instead assume that students have previous experience from the lecture class and the textbook. Since the lecture and textbook treatment of angular momentum tends to be highly mathematical (*e.g.,* ladder operators), the *Tutorials* focus heavily on building conceptual aspects of angular momentum.

D. Deep Similarities between the Curricula

Despite the overt curricular differences discussed above, we also observed some deep similarities between the theoretical commitments and the influence of those commitments on what each curricula tries to accomplish in the classroom. For example, social constructivism underlies at least one theoretical commitment for each group (P2 and T16). That is, both groups believe that knowledge is constructed by the students and that social interactions are critical to the construction of such knowledge.

Both curricula also value students expressing their knowledge in more than one way: the *Paradigms* with a very explicit focus on multiple representations and on students translating information between representations, the *Tutorials* on students articulating the meaning of mathematics and of physical concepts using words and reasoning.

A broader similarity that is not immediately apparent from the examples described here is that the developers of each curriculum take a "big-picture" perspective when designing activities. That is, we each think not only about the local learning goals for a particular activity, but also about how that activity fits into the broader sequence of experiences that we expect students to have over one or more courses. Part of the reason for taking such a big-picture view can be traced to the practical structures for each curriculum, but there are also strong indicators of the importance of thinking broadly in the theoretical commitments.

Although the *Paradigms* consists of an entire year-long sequence of junior-level courses, the individual courses are taught by separate instructors and so are not necessarily completely coordinated. Over the years, however, the various *Paradigms* instructors have made an effort (especially in quantum mechanics) to make use of certain activity structures and question types across the different quantum *Paradigms*. An example of this can be seen in the discussion of the Ring activity in Sec. III C 2, the structure of which is not only repeated throughout *Central Forces* but in the quantum courses that come before (*Quantum Fundamentals*) and after (the *Quantum Capstone*). This repeated structure supports students in making Individual Cognitive Connections (P1) by allowing them to revisit similar reasoning several times over the course of their junior and senior years in progressively more complex contexts.

The *Tutorials*, which take a more supplementary role, would not necessarily need to maintain cohesive themes across the quantum courses at UW. Each tutorial could focus on addressing student difficulties with one particular context or idea, with little to no coordination between tutorials from week to week. However, the *Tutorials* commitment to helping students develop a Coherent Framework (T13) pushed us to identify meta-goals that span the entire tutorial sequence. Early tutorials (given near the beginning of students' studies of QM) tend to focus on helping students identify and implement basic quantum rules, such as the probability postulate, while later tutorials remind students of these rules and help them learn the nuances of using them in more complicated physical scenarios. The example activity discussed in Sec. IV C is primarily an example of the latter, building on students' previously developed intuitions.

VI. MESSAGE FOR INSTRUCTORS

As there is an increasing demand for research-based instructional material for the teaching of quantum mechanics, we would like to address current and prospective instructors directly. The *Paradigms* and the *Tutorials* each represent an attempt to leverage research on student understanding, accumulated pedagogical content knowledge, and best practices in education to create activities to help students learn quantum mechanics. The curricula themselves look very different, and are each comprehensive enough that they can look intimidating to prospective adopters. The authors would like to forefront some of the observations discussed earlier in this paper that may be helpful to instructors who are interested in making use of materials like the *Paradigms* or the *Tutorials* but who may not know which to choose or where to start.

First, each curriculum is likely to be particularly easy to implement within a structural environment similar to the one for which it was designed. That is, the *Tutorials* work well for large classes with recitation sections (or similar 50-minute chunks of time), while the *Paradigms* may work better for smaller class sizes and can often be implemented in smaller time chunks. However, we note that each curriculum can be (and has been) adapted for other constraints. For example, the QM *Tutorials* have been given as interactive tutorial-lectures in classes with as many as 150 students. The actual implementation of *Paradigms* activities can vary substantially from instructor to instructor—they can be implemented flexibly if the instructor is willing to take active steps to ensure pieces continue to fit together as they are changed on the fly, or they can be given following a more proscribed structure.

Second, the different theoretical commitments that each curriculum prioritized may help instructors not only choose which activities to adopt but also understand aspects of their implementation more clearly. The *Paradigms* may be especially useful for instructors who value attending directly to their own students' ideas or who emphasize metacognition and self-reflection. In contrast, the *Tutorials* may be more helpful for instructors who value the construction of a coherent framework for quantum mechanics, or who think their students would benefit from more highly-structured materials. Despite these differences in focus, however, both groups share the attitude that teaching with research-based instructional materials should be done thoughtfully: try something out in the classroom, reflect carefully on what happens (and why), and refine it for next year.

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