3.4 Magnetic Resonance

Magnetic Resonance Imaging (MRI)

Modern 3 tesla clinical MRI scanner.

Magnetic Resonance Image showing a vertical cross section through a human head.

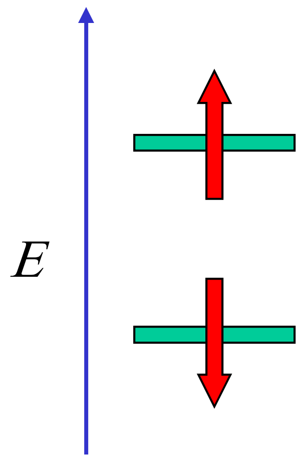
http://en.wikipedia.org/wiki/Magnetic_resonance_imaging

Uniform magnetic field: $\vec{B}_0 = B_0 \hat{z}$

$$H_0 = \omega_0 S_z , \quad \omega_0 \equiv \frac{eB_0}{m_e c}$$

$$H_0 \left| + \right\rangle = \frac{\hbar \omega_0}{2} \left| + \right\rangle$$

$$H_0 \left| - \right\rangle = -\frac{\hbar \omega_0}{2} \left| - \right\rangle$$

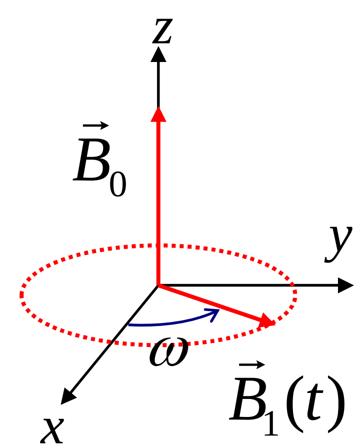


$$E_+ = \frac{\hbar \omega_0}{2}$$

$$E_- = -\frac{\hbar \omega_0}{2}$$

$$\Delta E = E_+ - E_- = \hbar \omega_0$$

Additional rotating magnetic field: $\vec{B}_1 = B_1 \cos \omega t \ \hat{x} + B_1 \sin \omega t \ \hat{y}$



Total magnetic field: $\vec{B} = \vec{B}_0 + \vec{B}_1 = B_0 \hat{z} + B_1 \left(\cos \omega t \ \hat{x} + \sin \omega t \ \hat{y} \right)$

Time dependent Hamiltonian:

$$H(t) = -\vec{\mu} \bullet \vec{B} = H_0 + H_1(t) = \omega_0 S_z + \omega_1 \left(\cos \omega t \, S_x + \sin \omega t \, S_y \right),$$

$$\omega_0 \equiv \frac{eB_0}{m_e c}, \quad \omega_1 \equiv \frac{eB_1}{m_e c}$$

Matrix representation of the Hamiltonian:

$$H(t) = \omega_0 S_z + \omega_1 \left(\cos\omega t\, S_x + \sin\omega t\, S_y\right)$$

$$\doteq \frac{\hbar}{2}\begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & \omega_1 \cos\omega t \\ \omega_1 \cos\omega t & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & -i\omega_1 \sin\omega t) \\ i\omega_1 \sin\omega t & 0 \end{pmatrix}$$

$$\doteq \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}$$

Schrödinger Equation : $i\hbar \frac{d}{dt}\left|\psi(t)\right\rangle = H(t)\left|\psi(t)\right\rangle, \ \left|\psi(t)\right\rangle = c_+(t)\left|+\right\rangle + c_-(t)\left|-\right\rangle \doteq \begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix}$

$$\begin{pmatrix} i\hbar \dfrac{dc_+(t)}{dt} \\ i\hbar \dfrac{dc_-(t)}{dt} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}\begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix}$$

$$\Rightarrow \begin{cases} i\hbar \dfrac{dc_+(t)}{dt} = \dfrac{\hbar\omega_0}{2} c_+(t) + \dfrac{\hbar\omega_1}{2} e^{-i\omega t} c_-(t) \\ i\hbar \dfrac{dc_-(t)}{dt} = \dfrac{\hbar\omega_1}{2} e^{i\omega t} c_+(t) - \dfrac{\hbar\omega_0}{2} c_-(t) \end{cases}$$

State vector as viewed from the rotation frame:

$$|\psi(t)\rangle \doteq \begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix} \longrightarrow |\tilde{\psi}(t)\rangle = R|\psi(t)\rangle \doteq \begin{pmatrix} e^{i\frac{\omega t}{2}} & 0 \\ 0 & e^{-i\frac{\omega t}{2}} \end{pmatrix} \begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix} = \begin{pmatrix} c_+(t)e^{i\frac{\omega t}{2}} \\ c_-(t)e^{-i\frac{\omega t}{2}} \end{pmatrix}$$

$$R(\omega t) \doteq \begin{pmatrix} e^{i\frac{\omega t}{2}} & 0 \\ 0 & e^{-i\frac{\omega t}{2}} \end{pmatrix}$$

$$= \alpha_+(t)|+\rangle + \alpha_-(t)|-\rangle \doteq \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix}, \qquad \begin{aligned} \alpha_+(t) &= c_+(t)e^{i\frac{\omega t}{2}} \\ \alpha_-(t) &= c_-(t)e^{-i\frac{\omega t}{2}} \end{aligned}$$

$$\Longrightarrow \quad |\psi(t)\rangle = \alpha_+(t)e^{-i\frac{\omega t}{2}}|+\rangle + \alpha_-(t)e^{i\frac{\omega t}{2}}|-\rangle \doteq \begin{pmatrix} \alpha_+(t)e^{-i\frac{\omega t}{2}} \\ \alpha_-(t)e^{i\frac{\omega t}{2}} \end{pmatrix}$$

Schrödinger Equation becomes:

$$i\hbar \frac{d\alpha_+(t)}{dt} e^{-i\frac{\omega}{2}t} + i\hbar\left(-i\frac{\omega}{2}\right)\alpha_+(t)e^{-i\frac{\omega}{2}t} = \frac{\hbar\omega_0}{2}\alpha_+(t)e^{-i\frac{\omega}{2}t} + \frac{\hbar\omega_1}{2}\alpha_-(t)e^{-i\frac{\omega}{2}t}$$

$$i\hbar \frac{d\alpha_-(t)}{dt} e^{i\frac{\omega}{2}t} + i\hbar\left(i\frac{\omega}{2}\right)\alpha_-(t)e^{i\frac{\omega}{2}t} = \frac{\hbar\omega_1}{2}\alpha_+(t)e^{i\frac{\omega}{2}t} - \frac{\hbar\omega_0}{2}\alpha_-(t)e^{i\frac{\omega}{2}t}$$

$$\left\{\begin{aligned} i\hbar\frac{d\alpha_+(t)}{dt} &= -\frac{\hbar\Delta\omega}{2}\alpha_+(t)+\frac{\hbar\omega_1}{2}\alpha_-(t) \\ i\hbar\frac{d\alpha_-(t)}{dt} &= \frac{\hbar\omega_1}{2}\alpha_+(t)+\frac{\hbar\Delta\omega}{2}\alpha_-(t) \end{aligned}\right.$$

where $\Delta\omega \equiv \omega - \omega_0$

$$\begin{pmatrix} i\hbar\dfrac{d\alpha_+(t)}{dt} \\ i\hbar\dfrac{d\alpha_-(t)}{dt} \end{pmatrix} = \begin{pmatrix} -\dfrac{\hbar\Delta\omega}{2} & \dfrac{\hbar\omega_1}{2} \\ \dfrac{\hbar\omega_1}{2} & \dfrac{\hbar\Delta\omega}{2} \end{pmatrix}\begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix}$$

$$\Rightarrow \quad i\hbar\frac{d}{dt}\begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -\Delta\omega & \omega_1 \\ \omega_1 & \Delta\omega \end{pmatrix}\begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix}$$

$$\Rightarrow \quad i\hbar\frac{d}{dt}\left|\tilde{\psi}(t)\right\rangle = \tilde{H}\left|\tilde{\psi}(t)\right\rangle, \quad \tilde{H} = \frac{\hbar}{2}\begin{pmatrix} -\Delta\omega & \omega_1 \\ \omega_1 & \Delta\omega \end{pmatrix}$$

In the rotating frame the Hamiltonian is time independent!!

When $\vec{B} = B_0\,\hat{z} + B_1\,\hat{x}$

$$H \doteq \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 \\ \omega_1 & -\omega_0 \end{pmatrix} \qquad \omega_0 = \frac{eB_0}{m_e c}, \quad \omega_1 = \frac{eB_1}{m_e c}$$

Rabi Formula $P(+\to-) = \dfrac{\omega_1^2}{\omega_0^2+\omega_1^2}\sin^2\left(\dfrac{\sqrt{\omega_0^2+\omega_1^2}}{2}t\right)$

$$P(+\to-) = \left|\langle -|\psi(t)\rangle\right|^2 = \left|c_-(t)\right|^2$$

$$= \left|e^{-i\frac{\omega}{2}t}\alpha_-(t)\right|^2 = \left|\alpha_-(t)\right|^2 = \left|\langle -|\tilde{\psi}(t)\rangle\right|^2$$

Rabi Formula with $\omega_0 \to -\Delta\omega$

$$P(+\to-) = \frac{\omega_1^2}{\Delta\omega^2+\omega_1^2}\sin^2\left(\frac{\sqrt{\Delta\omega^2+\omega_1^2}}{2}t\right)$$

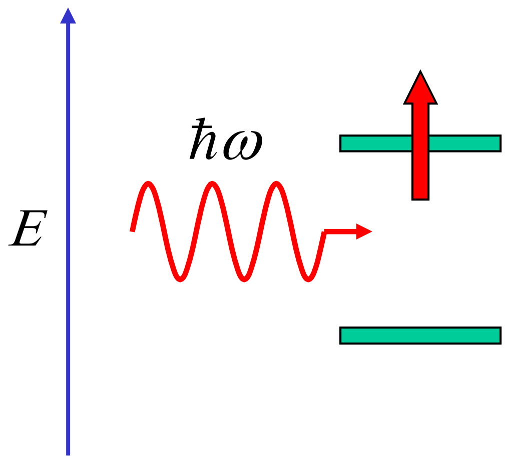
$$= \frac{\omega_1^2}{(\omega-\omega_0)^2+\omega_1^2}\sin^2\left(\frac{\sqrt{(\omega-\omega_0)^2+\omega_1^2}}{2}t\right)$$

Rabi flopping equation

$$P(+\rightarrow -)=\frac{\omega_1^2}{\Omega^2}\sin^2\left(\frac{\Omega}{2}t\right)\quad \text{where}\quad \Omega=\sqrt{(\omega-\omega_0)^2+\omega_1^2}$$

Generalized Rabi frequency

When $\omega=\omega_0$ (resonance condition), $\Omega=\omega_1=\dfrac{eB_1}{m_e c}$: Rabi frequency.



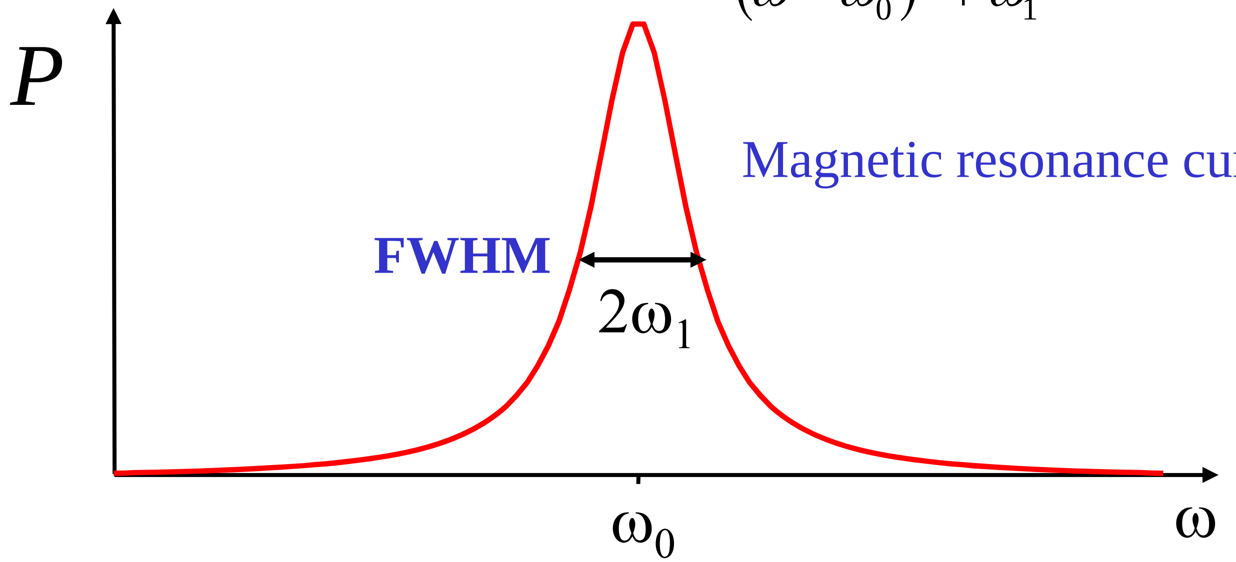
$$E_+=\frac{\hbar\omega_0}{2}$$

$$\Delta E=E_+-E_-=\hbar\omega_0$$

$$E_-=-\frac{\hbar\omega_0}{2}$$

$$P(+\rightarrow -)=\sin^2\left(\frac{\omega_1}{2}t\right)$$

Amplitude of the spin flip probability: $\dfrac{\omega_1^2}{(\omega-\omega_0)^2+\omega_1^2}$ **Lorentzian curve**



Time dependent spin flip probability: $P(+\rightarrow-)=\sin^2\left(\dfrac{\omega_1}{2}t\right)$