Electrostatic Potential Due to a Ring of Charge (Code:3D)

The problem I was asked to solve was to find the electrostatic potential due to a ring of charge. The ring had a radius R and a total charge Q.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r_i}|} \tag{1}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\vec{r'})|d\vec{r'}|}{|\vec{r}-\vec{r'}|} \tag{2}$$

$$\lambda = \frac{Q}{2\pi R} \tag{3}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi R}\int\frac{|d\vec{r'}|}{|\vec{r}-\vec{r'}|} \tag{4}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{|\vec{r}-\vec{r'}|} \tag{5}$$

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{\sqrt{(r^2+R^2+z^2-2rR\cos(\phi-\phi')}} \tag{6}$$

V along the z-axis:

$$V(r\!=\!0,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{\sqrt{R^2+z^2}} \tag{7}$$

$$V(r\!=\!0,\phi,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2+z^2}} \tag{8}$$

I was then prompted to expand this in a power series to approximate V at points very close to zero. After recognizing that I needed to use the power series

$$(1+c)^p = 1+pc+\frac{p(p-1)}{2!}c^2+... \tag{9}$$

$$V = \frac{Q}{4\pi\epsilon_0 R}\left(1+\frac{z^2}{R^2}\right)^{-\frac{1}{2}} \tag{10}$$

$$V(z) = \frac{Q}{4\pi\epsilon_0 R}\left(1 - \frac{z^2}{2R^2} + \frac{3z^4}{8R^4} + ...\right) \tag{11}$$

I discovered that unless I focused on a specific axis, the simplest form of an expression can came as an unsolveable integral. I probably would not have recognized this at first. I also discovered that changing the position vectors into rectangular coordinates and then describing each of their rectangular components in polar form can allow for easier manipulation. After focusing on the z-axis I saw that an otherwise difficult integral to calculate can become manageable. After expanding my solution in a power series that was familiar to me, I also saw that the electrostatic potential contained only even powers of z. After letting z approach infinity, the expression for potential became $V(z) = \frac{Q}{4\pi\epsilon_0 R}$ which is the potential due to a point charge. This makes sense because as you get further and further away, the ring appears to vanish to a single point.

The group that evaluated points far from 0 along the z-axis had an answer that was similar to mine, but with the z terms in the denominator and the R terms in the numerator. The group that had evaluated the expression on the x, y-plane at points close to 0 had the expression $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{R}\left[2\pi + \frac{2}{2R^2}\right]$. The group that evaluated it on this plane for points far outside the ring had a similar expression with the R's and r's swapped. If you look at all of these results collectively you will see that at points very far from the ring you approach the expression for electrostatic potential due to a point charge, and for the point at the center of the ring you get the electrostatic potential due to a point charge a distance R away with charge Q.