Quantum Calculations on a Ring I

Before you begin, recall that an arbitrary state $|\Phi\rangle$ can be written in the L_z eigenbasis as

$$|\Phi\rangle \doteq \begin{pmatrix} \vdots \\ \langle 2|\Phi\rangle \\ \langle 1|\Phi\rangle \\ \langle 0|\Phi\rangle \\ \langle -1|\Phi\rangle \\ \langle -2|\Phi\rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ a_2 \\ a_1 \\ a_0 \\ a_{-1} \\ a_{-2} \\ \vdots \end{pmatrix}$$

In this activity, your group will carry out calculaions on each of the following normalized abstract quantum states on a ring:

$$|\Phi_a\rangle = \sqrt{\frac{2}{12}}|3\rangle + \sqrt{\frac{1}{12}}|1\rangle + \sqrt{\frac{3}{12}}|0\rangle + \sqrt{\frac{2}{12}}|-1\rangle + \sqrt{\frac{1}{12}}|-3\rangle + \sqrt{\frac{3}{12}}|-4\rangle$$

$$|\Phi_b\rangle \doteq \begin{pmatrix} \vdots \\ 0 \\ \sqrt{\frac{2}{12}} \\ 0 \\ \sqrt{\frac{1}{12}} \\ \sqrt{\frac{3}{12}} \\ \sqrt{\frac{2}{12}} \\ 0 \\ \sqrt{\frac{1}{12}} \\ \sqrt{\frac{3}{12}} \\ \vdots \end{pmatrix}$$

$$\Phi_c(\phi) = \sqrt{\frac{1}{24\pi}}\left(\sqrt{2}\left(e^{i3\phi} + e^{-i\phi}\right) + \left(e^{i\phi} + e^{-i3\phi}\right) + \sqrt{3}(1 + e^{-i4\phi})\right)$$

For each question state the postulate(s) of quantum mechanics you use to complete the calculation and show explicitly how you use that postulate to answer the question.

1) If you measured the z-component of angular momentum for each state, what is the probability that you would obtain:

Group 1: $\hbar$	Group 2: $-4\hbar$	Group 3: $-3\hbar$	Group 4: $4\hbar$
Group 5: $-\hbar$	Group 6: 0	Group 7: $-2\hbar$	Group 8: $3\hbar$

2) If you measured the energy for each state, what is the probability that you would obtain:

Group 1: $\frac{9\hbar^2}{2I}$ Group 2: 0 Group 3: $\frac{4\hbar^2}{2I}$ Group 4: $\frac{\hbar^2}{2I}$

Group 5: $\frac{16\hbar^2}{2I}$ Group 6: $\frac{-\hbar^2}{2I}$ Group 7: $\frac{9\hbar^2}{2I}$ Group 8: $\frac{\hbar^2}{2I}$

3) How are the calculations you made for the different state representations similar and different? Be prepared to compare and contrast the calculations you made for each of the different representations (ket, matrix, wavefunction).

Solutions: For both questions 1 and 2, we expect the students to use Postulate 4. For questions three, we want them to simply say that in the for the first representation, they used bra-ket inner products, in the second representation they used matrix multiplication, and in the third, they carried out an integral. We want them to see that these are all three the same state and that the representationas are parallel and that, at least in principle, all the calculations can be carried out in each of the representations.

The table below is just a quick reference for the answers to questions 1 and 2.

Group	L_z	$P(L_z)$	E	P(E)
1	$\hbar$	1/12	$9\hbar^2/2I$	3/12
2	$-4\hbar$	3/12	0	3/12
3	$-3\hbar$	1/12	$4\hbar^2/2I$	0
4	$4\hbar$	0	$\hbar^2/2I$	3/12
5	$-\hbar$	2/12	$16\hbar^2/2I$	3/12
6	0	3/12	$-\hbar^2/2I$	0
7	$-2\hbar$	0	$9\hbar^2/2I$	3/12
8	$3\hbar$	2/12	$\hbar^2/2I$	3/12

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Quantum Calculations on a Ring II - Solutions

In this activity, your group will carry out calculations on the following normalized abstract quantum state on a ring:

$$|\Psi\rangle = \sqrt{\frac{1}{4}}\left(\sqrt{2}\,|1\rangle + |{-3}\rangle + |3\rangle\right)$$

1) You carry out a measurement to determine the energy of the particle at time t=0. Calculate the probability that you measure the energy to be $\frac{9\hbar^2}{2I}$. What representation/basis did you use to do this calculation and why did you use this representation?

Solutions: First we need to rewrite $|\psi\rangle$ with the time dependence shown explicitly.

$$|\Psi\rangle = \sqrt{\frac{1}{4}}\left(\sqrt{2}e^{\frac{i}{\hbar}\frac{\hbar^2}{2I}t}\,|1\rangle + e^{\frac{i}{\hbar}\frac{9\hbar^2}{2I}t}\,|{-3}\rangle + e^{\frac{i}{\hbar}\frac{9\hbar^2}{2I}t}\,|3\rangle\right)$$

Since both the $|-3\rangle$ and $|3\rangle$ states will result in an energy measurement of $\frac{9\hbar^2}{2I}$

$$P\left(E = \frac{9\hbar^2}{2I}\right) = |\langle 3|\psi\rangle|^2 + |\langle -3|\psi\rangle|^2$$

Substituting in for $|\psi\rangle$

$$\langle 3|\psi\rangle = \langle 3|\left(\sqrt{\frac{1}{4}}\left(\sqrt{2}e^{i\frac{\hbar}{2I}t}\,|1\rangle + e^{i\frac{9\hbar}{2I}t}\,|{-3}\rangle + e^{i\frac{9\hbar}{2I}t}\,|3\rangle\right)\right) = \sqrt{\frac{1}{4}}e^{i\frac{9\hbar}{2I}t}$$

Similarly,

$$\langle -3|\psi\rangle = \sqrt{\frac{1}{4}}e^{i\frac{9\hbar}{2I}t}$$

Thus,

$$P\left(E = \frac{9\hbar^2}{2I}\right) = \frac{1}{4}e^{-i\frac{9\hbar}{2I}t}e^{i\frac{9\hbar}{2I}t} + \frac{1}{4}e^{-i\frac{9\hbar}{2I}t}e^{i\frac{9\hbar}{2I}t} = \boxed{\frac{1}{2}}$$

2) You carry out a measurement on the location of the particle at time, t=0. Calculate the probability that the particle can be found in the region $0 < \phi < \frac{\pi}{3}$. What representation/basis did you use to do this calculation and why did you use this representation?

Solution: To carry out this calculation, we first need to write $|\psi\rangle$ in the (ϕ) basis using

$$\langle \phi, t|m\rangle = \frac{1}{\sqrt{2\pi}}e^{im\phi}e^{\frac{i}{\hbar}E_m t} = \frac{1}{\sqrt{2\pi}}e^{im\phi}e^{i\frac{m^2\hbar}{2I}t}$$

$$\psi(\phi, t) = \frac{1}{\sqrt{8\pi}}\left(\sqrt{2}e^{i\phi}e^{i\frac{\hbar}{2I}t} + e^{-i3\phi}e^{i\frac{9\hbar}{2I}t} + e^{i3\phi}e^{i\frac{9\hbar}{2I}t}\right)$$

To determine the probability of finding the particle between 0 and $\frac{\pi}{3}$, we must add up the probability of being in any position in this range by integrating

$$\int_0^{\frac{\pi}{3}} |\langle\phi, t|\psi\rangle|^2 d\phi = \int_0^{\frac{\pi}{3}} \psi(\phi,t)^*\psi(\phi,t)d\phi$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{8\pi}\left(\sqrt{2}e^{-i\left(\phi+\frac{\hbar}{2I}t\right)} + e^{i\left(3\phi-\frac{9\hbar}{2I}t\right)} + e^{-i\left(3\phi+\frac{9\hbar}{2I}t\right)}\right)$$
$$\left(\sqrt{2}e^{i\left(\phi+\frac{\hbar}{2I}t\right)} + e^{-i\left(3\phi-\frac{9\hbar}{2I}t\right)} + e^{i\left(3\phi+\frac{9\hbar}{2I}t\right)}\right) d\phi$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{8\pi}\left(2+1+1+\sqrt{2}\left(e^{-i\left(4\phi-\frac{8\hbar}{2I}t\right)} + e^{i\left(4\phi-\frac{8\hbar}{2I}t\right)}\right)\right.$$
$$\left.+\sqrt{2}\left(e^{i\left(2\phi+\frac{8\hbar}{2I}t\right)} + e^{-i\left(2\phi+\frac{8\hbar}{2I}t\right)}\right) + \left(e^{i6\phi} + e^{-i6\phi}\right)\right) d\phi$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{8\pi}\left(4 + 2\sqrt{2}\cos\left(4\phi - \frac{8\hbar}{2I}t\right) + 2\sqrt{2}\cos\left(2\phi + \frac{8\hbar}{2I}t\right) + 2\cos\left(6\phi\right)\right) d\phi$$

at $t = 0$ this simplifies to

$$\int_0^{\frac{\pi}{3}} |\langle\phi, t|\psi\rangle|^2 d\phi = \int_0^{\frac{\pi}{3}} \frac{1}{8\pi}\left(4 + 2\sqrt{2}\cos\left(4\phi\right) + 2\sqrt{2}\cos\left(2\phi\right) + 2\cos\left(6\phi\right)\right) d\phi$$

Carrying out the integrals, we get

$$\int_0^{\frac{\pi}{3}} |\langle\phi, t|\psi\rangle|^2 d\phi = \frac{1}{8\pi}\left(\frac{4\pi}{3} + 2\sqrt{2}\frac{1}{4}\sin\left(4\frac{\pi}{3}\right) + 2\sqrt{2}\frac{1}{2}\sin\left(2\frac{\pi}{3}\right) + \frac{1}{6}\sin\left(6\frac{\pi}{3}\right)\right)$$

$$\int_0^{\frac{\pi}{3}} |\langle\phi, t|\psi\rangle|^2 d\phi = \boxed{\frac{1}{6} - \frac{\sqrt{6}}{32\pi}}$$

3) You carry out a measurement to determine the energy of the particle at time $t = \frac{2I}{\hbar}\frac{\pi}{4}$. Calculate the probability that you measure the energy to be $\frac{9\hbar^2}{2I}$. What representation/basis did you use to do this calculation and why did you use this representation?
Solution: Looking at the solution to question 1, we see that the time factor squares away, so this quantity is independent of time.

4) You carry out a measurement on the location of the particle at time, $t = \frac{2I}{\hbar}\frac{\pi}{16}$. Calculate the probability that the particle can be found in the region $0 < \phi < \frac{\pi}{3}$. What representation/basis did you use to do this calculation and why did you use this representation?
Solution: At $t = \frac{2I}{\hbar}\frac{\pi}{16}$ this simplifies to

$$= \int_0^{\frac{\pi}{3}} \frac{1}{8\pi}\left(4 + 2\sqrt{2}\cos\left(4\phi - \frac{\pi}{2}\right) + 2\sqrt{2}\cos\left(2\phi + \frac{\pi}{2}\right) + 2\cos\left(6\phi\right)\right) d\phi$$

Carrying out the integrals, we get

$$= \frac{1}{8\pi}\left(\frac{4\pi}{3} + 2\sqrt{2}\frac{1}{4}\sin\left(4\phi - \frac{\pi}{2}\right)|_0^{\frac{\pi}{3}} + 2\sqrt{2}\frac{1}{2}\sin(2\phi)\,|_0^{\frac{\pi}{3}} + \frac{1}{6}\sin(6\phi)\,|_0^{\frac{\pi}{3}}\right)$$

$$\int_0^{\frac{\pi}{3}} |\langle\phi, t|\psi\rangle|^2 d\phi = \frac{1}{8\pi}\left(\frac{4\pi}{3} + \frac{\sqrt{2}}{2}\frac{3}{2} + \sqrt{2}\frac{-3}{2}\right) = \boxed{\frac{1}{6} - \frac{3\sqrt{2}}{32\pi}}$$

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Quantum Calculations on a Ring III

Consider the following normalized abstract quantum state on a ring:

$$\Phi(\phi) = \sqrt{\frac{8}{5\pi}} \cos^3(2\phi)$$

1) If you measured the z-component of angular momentum, what is the probability that you would obtain $2\hbar$? $-3\hbar$?

Solution: To simplify our calculations, we can rewrite $\Phi(\phi)$ in terms of the eigenstates of the hamiltonian, $\langle\phi|m\rangle = \frac{1}{\sqrt{2\pi}} e^{im\phi}$

$$\Phi(\phi) = \sqrt{\frac{8}{5\pi}} \left(\frac{1}{2}\left(e^{i2\phi} + e^{-i2\phi}\right)\right)^3$$

$$= \sqrt{\frac{8}{5\pi}} \frac{1}{8} \left(e^{i2\phi} + e^{-i2\phi}\right)\left(e^{i2\phi} + e^{-i2\phi}\right)\left(e^{i2\phi} + e^{-i2\phi}\right)$$

$$= \sqrt{\frac{1}{20}} \sqrt{\frac{1}{2\pi}} \left(e^{i6\phi} + e^{-i6\phi} + 3e^{i2\phi} + 3e^{-i2\phi}\right)$$

Thus,

$$\Phi(\phi) = \sqrt{\frac{1}{20}} \left(\langle\phi|6\rangle + \langle\phi|-6\rangle + 3\langle\phi|2\rangle + 3\langle\phi|-2\rangle\right)$$

and

$$\boxed{|\Phi\rangle = \sqrt{\frac{1}{20}} \left(|6\rangle + |-6\rangle + 3|2\rangle + 3|-2\rangle\right)}$$

From here it is clear that the probability of measuring $2\hbar$ is $\frac{9}{20}$ and the probability of measuring $-3\hbar$ is zero.

Note, while in this case it is relatively easy to extract the coefficients for the eigenstates algebraically, one can obtain the coefficients for each eigenstate in the eigenstate expansion

$$|\Phi\rangle = \sum_m a_m |m\rangle$$

in the general case by taking the inner product of $|\Phi\rangle$ with each eigenstate

$$a_m = \langle m|\Phi\rangle = \int_0^{2\pi} \langle m|\phi\rangle\langle\phi|\Phi\rangle d\phi$$

$$= \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{-im\phi} \sqrt{\frac{8}{5\pi}} \cos^3(2\phi) d\phi$$

This integral can be evaluated using Maple or with an integral table.

2) If you measured the z-component of angular momentum, what other possible values could you obtain with non-zero probability?

Solution: From the state $|\Phi\rangle$ found above it is clear that one could measure $2\hbar$, $-2\hbar$, $6\hbar$, and $-6\hbar$

3) If you measured the energy, what possible values could you obtain with non-zero probability?

Solution: From the state $|\Phi\rangle$ found above it is clear that one could measure $\frac{4\hbar^2}{2I}$ with a probability of $\frac{18}{20}$ and $\frac{36\hbar^2}{2I}$ with a probability of $\frac{2}{20}$

4) What is the probability that the particle can be found in the region $0 < \phi < \frac{\pi}{2}$? Solution: To find the probability of finding the particle in the region $0 < \phi < \frac{\pi}{2}$ we must work in the position representation. The probability is then

$$P\left(0 < \phi < \frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \Phi(\phi)^* \Phi(\phi) d\phi$$

$$P\left(0 < \phi < \frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \left(\sqrt{\frac{8}{5\pi}} \cos^3(2\phi)\right)^2 d\phi$$

$$= \frac{8}{5\pi} \int_0^{\frac{\pi}{2}} \cos^6(2\phi) d\phi$$

$$= \frac{8}{5\pi} \int_0^{\frac{\pi}{2}} \cos^6(2\phi) d\phi$$

Using Maple or a Table of Integrals,

$$= \frac{8}{5\pi} \left(\frac{1}{12} \cos(2\phi)^5 \sin(2\phi) + \frac{5}{48} \cos(2\phi)^3 \sin(2\phi) + \frac{5}{32} \cos(2\phi) \sin(2\phi) + \frac{5}{16}\phi\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$P\left(0 < \phi < \frac{\pi}{2}\right) = \frac{1}{4}$$

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