# 1 Physical constants

$$\begin{aligned}
\text{fine structure constant :}\quad & \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} \\
\text{Rydberg energy :}\quad & E_o = \frac{m_e e^4}{2\hbar^2} \\
& \quad = \frac{m_e c^2 \alpha^2}{2} \\
\text{Bohr magneton :}\quad & \mu_B = \frac{e\hbar}{2m_e c} \\
\text{Bohr radius :}\quad & a_o = \frac{\hbar^2}{m_e e^2}
\end{aligned}$$

# 2 Vector calculus relationships

Triple products:

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \\
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})
\end{aligned}$$

Product rules:

$$\begin{aligned}
\nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \\
&\quad + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \\
\nabla \cdot (\phi \mathbf{A}) &= \phi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \phi \\
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) + \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + \\
&\quad + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}
\end{aligned}$$

Second derivatives:

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\
\nabla \cdot (\nabla \times \mathbf{A}) &= 0
\end{aligned}$$

Green's theorem:

$$\int_V \left(\psi \nabla^2 \phi - \phi \nabla^2 \psi\right) dV = \oint_S \left(\psi \nabla \phi - \phi \nabla \psi\right) \cdot \mathbf{dS}$$

Spherical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \\
\nabla \times \mathbf{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial \phi}\right)\hat{\mathbf{r}} \\
&\quad + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r A_\phi\right)\right)\hat{\theta} \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r A_\theta\right) - \frac{\partial A_r}{\partial \theta}\right)\hat{\phi} \\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) \\
&\quad + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}
\end{aligned}$$

Cylindrical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\
\nabla \cdot \mathbf{A} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} \\
&\quad + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\phi} \\
&\quad + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\mathbf{z}} \\
\nabla^2 f &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

# 3 Quantum mechanics

Raising and lowering operators for ang. momentum:

$$\begin{aligned}
J_\pm &= J_x \pm i J_y \\
J_\pm \left|j, m\right\rangle &= \hbar\sqrt{j(j+1) - m(m \pm 1)}\left|j, m \pm 1\right\rangle
\end{aligned}$$

Perturbation theory for nondegenerate states:

$$E_n \approx E_n^o + \langle n|\, V \,|n\rangle + \sum_{m \neq n} \frac{\left|\langle n|\, V \,|m\rangle\right|^2}{E_n - E_m} + \cdots$$

Harmonic oscillator: $[a, a^\dagger] = 1$

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\omega\hbar}}$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\omega\hbar}}$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$a|n\rangle = \sqrt{n}|n-1\rangle$$

# 4 Electromagnetism

Maxwell's equations:

$$\nabla\cdot\mathbf{D} = 4\pi\rho \qquad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}$$

Magnetic dipole field:

$$\mathbf{B}(\mathbf{r}) = \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{r^3}$$

Energy density: $U = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$
Poynting vector: $\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$

## General solutions of Laplace's equation

in cylindrical coordinates (independent of $z$):

$$\Phi(\rho,\phi) = a_o\log(\rho) + \sum_{n=1}^{\infty}\left(\frac{a_n}{\rho^n} + b_n\rho^n\right)(c_n\cos n\phi + d_n\sin n\phi)$$

in spherical coordinates:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)$$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

(with azimuthal symmetry)

# 5 Useful math formulas

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$
$$\int_{-\infty}^{\infty} e^{ixy}dy = 2\pi\delta(x)$$
$$\int_0^{\infty} x^n e^{-x}dx = n!\,,\ \text{integer } n$$

$$(1+x)^n = \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}x^k$$
$$\log(n!) \approx \frac{1}{2}\log(2\pi n) + n\log(n) - n$$
$$\sin(x\pm y) = \sin x\cos y \pm \cos x\sin y$$
$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$\frac{1}{|\mathbf{x}-r'\hat{\mathbf{z}}|} = \sum_l \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

Spherical Bessel functions:

$$j_0(z) = \frac{\sin z}{z} \qquad n_0(z) = -\frac{\cos z}{z}$$
$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \qquad n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$

Legendre polynomials:

$$P_0(x) = 1 \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$
$$P_1(x) = x \qquad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$
$$P_l^m(x) = \left(1 - x^2\right)^{m/2}\frac{d^m P_l}{dx^m}$$

Spherical harmonics:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{i2\phi}$$
$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \qquad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$