QM II
Phys 652

# Lecture # 8

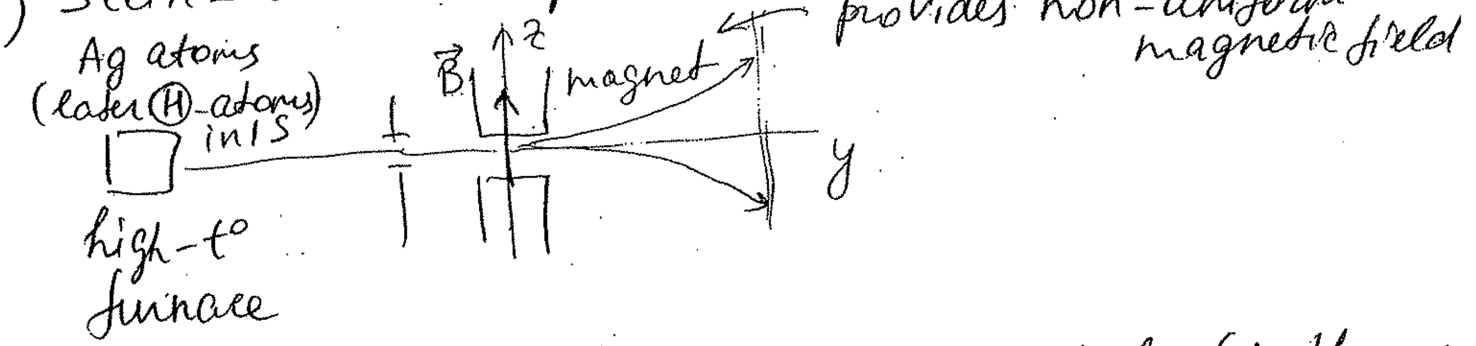
## Spin

1925: Goudsmit & Uhlenbeck hypothesis:

Every electron has an intrinsic angular momentum (spin) of $\frac{\hbar}{2}$, which corresponds to a magnetic moment of one Bohr magneton $\mu_B = \frac{q\hbar}{2m_e}$

This hypothesis was based on several experimental facts:

1) Stern - Gerlach experiment



Ⓗ-atom in magnetic fields is described by (if the spin is not taken into account)

$$H = \underbrace{H_0}_{\uparrow} + \underbrace{H_1}_{\uparrow} + \underbrace{H_2} = \frac{q^2\vec{B}^2}{8m_e}(x^2+y^2) \leftarrow \text{diamagnetic term}$$

$$\frac{\vec{P}^2}{2m_e} - \frac{e^2}{r} \qquad -\vec{M}\cdot\vec{B} \leftarrow \text{paramagnetic term}$$

$$\vec{M} = \frac{q}{2m_e}\vec{L}$$

↑ orbital angular momentum

⇓ typically much smaller than paramagnetic

The force acting on our atoms in a non-uniform magnetic field is $\vec{F} = -\vec{\nabla}(-\vec{M}\cdot\vec{B}) \simeq M_z \vec{\nabla} B_{(z)}$

Experimental result; beam splits into two components ⇒ $M_z$ can assume two values

⇑ where from?

If Ⓗ-atoms are in 1s state ⇒ $l=0 \Rightarrow L_z=0 \Rightarrow M_z=0$

(no splitting expected)

⇓ must be something else

⇓

spin $\vec{S} \Rightarrow \vec{M}_s = \frac{q}{m_e}\vec{S}$

(in general, $\vec{M} = g\left(\frac{q}{2m_e}\right)\vec{J}$)

↑ gyromagnetic ratio

⇓

if $S_z|\psi\rangle = \hbar m_s|\psi\rangle$

$m_s = \pm\frac{1}{2} \Rightarrow S_z = \pm\frac{\hbar}{2}$

## 2) Anomalous Zeeman effect

Consider Ⓗ-atom in a magnetic field (with no spin) ⇒

$(H_0 + H_1)|n, l, m\rangle = ?$ Since $H_1 = -\vec{M}\cdot\vec{B} =$

$= -\frac{\mu_B}{\hbar}\vec{L}\cdot\vec{B} =$ ↖ $B \parallel Oz$

$= -\frac{\mu_B}{\hbar}L_z B$

$[H_0, H_1] = 0$ ⇐

(because $[H_0, L_z] = 0$)

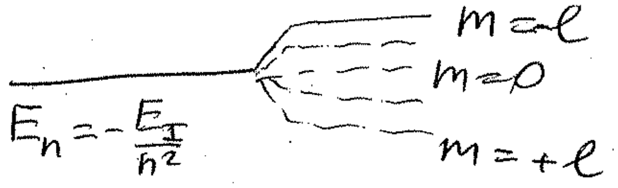
⇓ then $H_0$ & $H_1$ share the same eigenstates ⇒

$$\left(H_0 - \frac{\mu_B}{\hbar}BL_z\right)|n, l, m\rangle = \left(E_n - \frac{\mu_B}{\cancel{\hbar}}B\cdot\cancel{\hbar}m\right)|n, l, m\rangle$$

⇒

Energy in a B-field is $\underline{E_n - \mu_B B m}$ ← degeneracy with respect to m is removed

This means that the n th level which is normally $n^2$-degenerate



$E = E_n - \mu_B B m$

splits into $2l+1)$-sublevels

⇓

Zeeman effect

Since $l$ is an integer ⇒ supposed to get an <u>odd</u> number of levels

Experiment : <u>even</u> number → "anomalous" Zeeman effect
in Ⓗ atom

⇓

introduce spin degrees of freedom

⇓

The system is described by a set

$$\{\underbrace{H, \vec{L}^2, L_z}_{\uparrow\ \text{external (orbital) degrees of freedom}}, \underbrace{\vec{S}^2, S_z}_{\uparrow\ \text{spin degrees of freedom}}\}$$

<u>Properties of spin operators</u>

1) $\vec{S}$ is an angular momentum

⇓

general rules apply, e.g. $[S_i, S_j] = i\hbar\,\varepsilon_{ijk} S_k$

2) Similar to the orbital angular momentum

$$\vec{L}^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle \qquad \vec{S}^2|s,m_s\rangle = \hbar^2 s(s+1)|s,m_s\rangle$$

$$L_z|l,m\rangle = \hbar m|l,m\rangle \quad \Leftrightarrow \quad S_z|s,m_s\rangle = \hbar m_s|s,m_s\rangle$$

$$|m| \le l \qquad\qquad |m_s| \le s$$

A given particle is characterized by a unique value of $s$

3) All spin observables commute with all orbital observables, e.g. $[\vec{L}^2, \vec{S}^2] = 0$, etc.

4) In the case of the electron $s = \frac{1}{2} \Rightarrow$ there are $2s+1 = 2$ spin degrees of freedom $\Rightarrow$ the spin states are

$$|+\rangle \equiv |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$
$$|-\rangle \equiv |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

Then, $\vec{S}^2|\pm\rangle = \hbar^2 \cdot \underset{\substack{\uparrow \\ s(s+1)}}{\frac{3}{4}}|\pm\rangle$ ; $\overset{s}{S_z}|\overset{m_s}{\pm}\rangle = \pm\frac{\hbar}{2}|\pm\rangle$

- Orthonormality:

$$\langle +|-\rangle = \langle -|+\rangle = 0; \quad \langle +|+\rangle = \langle -|-\rangle = 1$$

- Closure:

$$|+\rangle\langle +| + |-\rangle\langle -| = 1$$

- Projection operators $P_+ = |+\rangle\langle +|$
  $P_- = |-\rangle\langle -|$

- Raising & lowering operators

$$S_\pm = S_x \pm iS_y \Rightarrow S_\pm|\pm\rangle = 0$$

$$\text{Recall} \nearrow \begin{cases} S_-|+\rangle = \hbar|-\rangle \\ S_+|-\rangle = \hbar|+\rangle \end{cases}$$

$$J_\pm|j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$

($j = \frac{1}{2}$, $m \equiv m_s = \pm\frac{1}{2}$ for $|\pm\rangle$ in this case)

- Matrix representations

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_{+}, \quad |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_{-}$$

Then, an arbitrary spin state would be described as $|\alpha\rangle = P_+|\alpha\rangle + P_-|\alpha\rangle = |+\rangle\langle+|\alpha\rangle +$

$$+ |-\rangle\langle-|\alpha\rangle \doteq \begin{pmatrix} \langle+|\alpha\rangle \\ \langle-|\alpha\rangle \end{pmatrix} \equiv \begin{pmatrix} C_+ \\ C_- \end{pmatrix} \leftarrow \text{2-component spinor} = C_+\chi_+ + C_-\chi_-$$

Similarly, $\langle\alpha| \doteq (\langle\alpha|+\rangle, \langle\alpha|-\rangle) =$

$$= (C_+^*, C_-^*)$$

Any operator in the $\{|+\rangle, |-\rangle\}$ basis can be represented by $2\times2$ matrix $\Rightarrow$

Ex. $S_z|\pm\rangle = \hbar \cdot (\pm\frac{1}{2})|\pm\rangle$

$$\Vert$$

$$S_z \doteq \begin{array}{ccc} |+\rangle & |-\rangle & \\ \hbar/2 & 0 & \langle+| \\ 0 & -\frac{\hbar}{2} & \langle-| \end{array} \textcircled{=}$$

$\langle+|S_z|+\rangle = \frac{\hbar}{2}$

$\langle-|S_z|-\rangle = -\frac{\hbar}{2}$

$\langle\pm|S_z|\mp\rangle = 0$

$$\textcircled{=} \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Similarly, $S_+ \doteq \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $S_- \doteq \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

It is customary to introduce <u>Pauli matrices</u>

$$\hat{\vec{S}} = \frac{\hbar}{2}\underset{\substack{\uparrow \\ \text{Pauli} \\ \text{matrices}}}{\hat{\vec{\sigma}}} , \quad \hat{\sigma}_x \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

## Properties of the Pauli matrices

- $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \underset{\nwarrow \text{identity matrix}}{1}$
- $\sigma_i \sigma_j + \sigma_j \sigma_i = 0 \quad (i \neq j)$
- $[\sigma_i , \sigma_j] = 2i\, \varepsilon_{ijk}\, \sigma_k$
- $\sigma_i^{+} = \sigma_i$ ← Hermitian
- $\det \sigma_i = -1$
- $\mathrm{Tr}\, \sigma_i = 0$

HW: prove that $(\hat{\vec{\sigma}} \cdot \vec{a})(\hat{\vec{\sigma}} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\, \hat{\vec{\sigma}} \cdot (\vec{a} \times \vec{b})$

Reading: Sakurai: pp. 163-165

pp. 26-29

# Examples of operations with Pauli matrices

1) Express $(2I+\sigma_x)^{-1}$ as a linear combination of the Pauli matrices and $I$.

Solution;

$$(2I+\sigma_x)^{-1} \underset{\substack{\uparrow \\ \frac{1}{1+x}=1-x+x^2-x^3+\dots}}{=} \frac{1}{2}\left(I+\frac{\sigma_x}{2}\right)^{-1} = \frac{1}{2}\left\{I-\frac{\sigma_x}{2}+\left(\frac{\sigma_x}{2}\right)^2-\right.$$

$$\left.-\left(\frac{\sigma_x}{2}\right)^3+\dots\right\} \underset{\substack{\uparrow \\ \sigma_i^2=I}}{=} \frac{1}{2}\left\{I-\frac{\sigma_x}{2}+\frac{I}{2^2}-\frac{1}{8}\sigma_x+\dots\right\}=$$

$$=\frac{1}{2}\left\{I\underbrace{\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\dots\right)}_{\shortparallel\ \frac{1}{1-\frac{1}{4}}=\frac{4}{3}}-\frac{\sigma_x}{2}\underbrace{\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\dots\right)}_{\shortparallel\ \frac{4}{3}}\right\}=$$

$$\underset{\substack{\uparrow \\ \text{geometric progression } \sum_{n=0}^{\infty} q^n=\frac{1}{1-q} \\ \text{for } |q|<1}}{=} \frac{1}{2}\cdot\frac{4}{3}\left(I-\frac{\sigma_x}{2}\right)=\boxed{\frac{2}{3}I-\frac{1}{3}\sigma_x}$$

2) Expand $\sigma_x^{-1}$ in terms of the Pauli matrices

Consider $\vec{n} = (1, 0, 0) \Rightarrow$

$$\underline{e^{-i\frac{\varphi}{2}\vec{\sigma}\cdot\vec{n}}} = \underline{e^{-i\frac{\varphi}{2}\sigma_x}} = \underline{\cos\frac{\varphi}{2} - i\sigma_x \sin\frac{\varphi}{2}}$$

Then, if $\varphi = \pi \Rightarrow e^{-i\frac{\pi}{2}\sigma_x} = 0 - i\sigma_x = -i\sigma_x$

so $\underline{\sigma_x = i e^{-i\frac{\pi}{2}\sigma_x}} \Rightarrow$

$$\sigma_x^{-1} = -ie^{i\frac{\pi}{2}\sigma_x} = -i\left(\underbrace{\cos\frac{\pi}{2}}_{\overset{\shortparallel}{0}}\cancel{\sigma_x} + i\underbrace{\sin\frac{\pi}{2}}_{\overset{\shortparallel}{1}}\sigma_x\right) ⊜$$

$$⊜\ \underline{\underline{\sigma_x}} \Rightarrow \boxed{\sigma_x^{-1} = \sigma_x}$$

aka Direct products → **Tensor products**

As we know, each quantum state of a particle is characterized by a state vector $|\Psi\rangle \in \mathcal{E}$

↑ abstract space (sub-space of a Hilbert space) ⇑ called "state space"

Recall Phys 657 ⟹

If $|\Psi\rangle$ describes a state of one particle in a 1D space $\Rightarrow |\Psi\rangle \in \mathcal{E}_x$

If same in a 3D space $\Rightarrow |\Psi\rangle \in \mathcal{E}_{\vec{r}}$

What is the relationship between $\mathcal{E}_x$ and $\mathcal{E}_{\vec{r}}$?

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Consider $\Psi(\vec{r}) = \underbrace{\Psi_x(x)}_{\in \mathcal{E}_x} \underbrace{\Psi_y(y)}_{\in \mathcal{E}_y} \underbrace{\Psi_z(z)}_{\in \mathcal{E}_z} \quad \in \mathcal{E}_{\vec{r}}$

⇓ present this in terms of state vectors ⟹

$$|\Psi\rangle = |\Psi_x\rangle \otimes |\Psi_y\rangle \otimes |\Psi_z\rangle$$

$\in \mathcal{E}_{\vec{r}}$, where $\mathcal{E}_{\vec{r}} = \mathcal{E}_x \underset{\substack{\uparrow \\ \text{tensor product}}}{\otimes} \mathcal{E}_y \otimes \mathcal{E}_z$

The vector space $\mathcal{E}$ is called the <u>tensor product</u> of $\mathcal{E}_1$ and $\mathcal{E}_2$, i.e. $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$, <u>if</u> there is a vector $|\varphi_1\rangle \otimes |\chi_2\rangle \in \mathcal{E}$ associated with each pair of vectors $|\varphi_1\rangle \in \mathcal{E}_1$ and $|\chi_2\rangle \in \mathcal{E}_2$ satisfying the following conditions:

$\uparrow$ Belong to space $\mathcal{E}_1$

- It is <u>linear</u> with respect to multiplication by complex numbers.

$$(\lambda |\varphi_1\rangle) \otimes |\chi_2\rangle = \lambda (|\varphi_1\rangle \otimes |\chi_2\rangle)$$

$$|\varphi_1\rangle \otimes (\mu |\chi_2\rangle) = \mu (|\varphi_1\rangle \otimes |\chi_2\rangle)$$

- It is <u>distributive</u> with respect to vector addition

$$|\varphi_1\rangle \otimes (|\chi_2\rangle + |\underset{\uparrow}{\varkappa_2}\rangle) = |\varphi_1\rangle \otimes |\chi_2\rangle + |\varphi_1\rangle \otimes |\varkappa_2\rangle$$

another vector from space $\mathcal{E}_2$

from space $\mathcal{E}_1$ $\swarrow$

$$(|\varphi_1\rangle + |\xi_1\rangle) \otimes |\chi_2\rangle = |\varphi_1\rangle \otimes |\chi_2\rangle + |\xi_1\rangle \otimes |\chi_2\rangle$$

- When a basis has been chosen in each of the spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ (e.g. $\{|u_{\underset{\uparrow}{i},\underset{\uparrow}{1}}\rangle\}$ in $\mathcal{E}_1$ and

\# of the vector    space $\mathcal{E}_1$

$\{|v_{j,2}\rangle\}$ in $\mathcal{E}_2$), the set of vectors

$$\{|u_{i,1}\rangle \otimes |v_{j,2}\rangle\}$$ constitutes a basis in $\mathcal{E}$

If $N_1$ and $N_2$ are dimensions of $\mathcal{E}_1$ and $\mathcal{E}_2$, the dimension of $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is $\underline{N = N_1 N_2}$

Note: the indexes 1 & 2 in $\mathcal{E}_1, \mathcal{E}_2$ may be referred to x and y, particles 1 and 2, etc.

Example Before we introduced spin, we described the Ⓗ-atom states as $|n,l,m\rangle \in \underset{\substack{\uparrow \\ \infty\text{-dimensional}}}{\mathcal{E}_{\vec{r}}}$

Introduction of spin requires spin variables ⇙

$|s, m_s\rangle \in \mathcal{E}_s$ ← for a given particle (2s+1)-dimensional

Altogether ⇒ $\underset{\substack{\in \\ \mathcal{E} = \mathcal{E}_{\vec{r}} \otimes \mathcal{E}_s}}{|n,l,m;s,m_s\rangle} = |n,l,m\rangle \otimes |s,m_s\rangle$

Theorem Operators acting in different spaces commute

Proof Consider arbitrary operators $A_1$ and $B_2$ acting in the spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively.

Let's act on a state vector $\underset{\substack{\in \\ \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2}}{|\psi\rangle} = \underset{\substack{\in \\ \mathcal{E}_1}}{|u_1\rangle} \otimes \underset{\substack{\in \\ \mathcal{E}_2}}{|v_2\rangle}$

To ensure proper dimensionality ⇓

present $A_1$ and $B_2$ as $A_1 \otimes \underset{\substack{\uparrow \\ \text{identity} \\ \text{in space } \mathcal{E}_2}}{I_2}$ and $\underset{\substack{\uparrow \\ \text{in space } \mathcal{E}_1}}{I_1} \otimes B_2$

Then,

$$[A_1 \otimes I_2, I_1 \otimes B_2]\,|\psi\rangle = A_1 \otimes I_2 I_1 \otimes B_2 |\psi\rangle -$$

$$- I_1 \otimes B_2 A_1 \otimes I_2 |\Psi\rangle = A_1 \otimes I_2 \underbrace{I_1 |u_1\rangle}_{\overset{\shortparallel}{|u_1\rangle}} \otimes B_2 |v_2\rangle -$$

$$- I_1 \otimes B_2 A_1 |u_1\rangle \otimes \underbrace{I_2 |v_2\rangle}_{\overset{\shortparallel}{|v_2\rangle}} = A_1 \otimes I_2 |u_1\rangle \otimes B_2 |v_2\rangle$$

$$- I_1 \otimes B_2 A_1 |u_1\rangle \otimes |v_2\rangle \overset{\text{see HW \#8(6)} \swarrow}{\doteq} A_1 |u_1\rangle \otimes B_2 |v_2\rangle -$$

$$- A_1 |u_1\rangle \otimes B_2 |v_2\rangle = 0 \Rightarrow \underline{[A_1 \otimes I_2, I_1 \otimes B_2] = 0}$$

## Examples of tensor product calculation

Consider basis vectors $|u_1\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|u_2\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of the basis $\{|u_i\rangle\} \in \mathcal{E}_u \Leftrightarrow \underset{(u)}{N_1} = 2$ ← dimensional of the

and $|v_1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ; $|v_2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ; $|v_3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

of the basis $\{|v_j\rangle\} \in \mathcal{E}_v$ with $N_2 = 3$

$\underline{\text{What is } |u_i\rangle \otimes |v_j\rangle \in \mathcal{E} = \mathcal{E}_u \otimes \mathcal{E}_v \text{ ?}}$

$$\Downarrow$$

$N = N_1 N_2 = 2.3 = 6$ ← dimensionality of $\mathcal{E}$

The new basis vectors are $|u_1\rangle \otimes |v_1\rangle$, $|u_1\rangle \otimes |v_2\rangle$, $|u_1\rangle \otimes |v_3\rangle$, $|u_2\rangle \otimes |v_1\rangle$, $|u_2\rangle \otimes |v_2\rangle$, $|u_2\rangle \otimes |$

Let's write them out in terms of six-$\underbrace{\text{component vectors}}_{\text{column}}$ ⇒

$$|u_1\rangle \otimes |v_1\rangle \doteq \left(\begin{array}{c}1\\0\\0\\\hline 0\\0\\0\end{array}\right) ; \quad |u_1\rangle \otimes |v_2\rangle \doteq \left(\begin{array}{c}0\\1\\0\\\hline 0\\0\\0\end{array}\right) ;$$

$$|u_1\rangle \otimes |v_3\rangle \doteq \left(\begin{array}{c}0\\0\\1\\\hline 0\\0\\0\end{array}\right) ; \quad |u_2\rangle \otimes |v_1\rangle \doteq \left(\begin{array}{c}0\\0\\0\\\hline 1\\0\\0\end{array}\right) ;$$

$$|u_2\rangle \otimes |v_2\rangle \doteq \left(\begin{array}{c}0\\0\\0\\\hline 0\\1\\0\end{array}\right) ; \quad |u_2\rangle \otimes |v_3\rangle \doteq \left(\begin{array}{c}0\\0\\0\\\hline 0\\0\\1\end{array}\right)$$

<u>Note:</u> for arbitrary 2D vectors $|u\rangle$ and 3D $|v\rangle$ $\Rightarrow$

$$|u\rangle \otimes |v\rangle \doteq \left(\begin{array}{c}u_1 v_1\\u_1 v_2\\u_1 v_3\\\hline u_2 v_1\\u_2 v_2\\u_2 v_3\end{array}\right)$$

What if we take matrices instead of the vectors?

Consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ ; $I = \begin{pmatrix} 1 & \overset{\swarrow}{0} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$(N_1 = 2)$ $\qquad\qquad$ $(N_2 = 3)$

$$A \otimes I = \left(\begin{array}{ccc|ccc} a_{11} & 0 & 0 & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} \\ \hline a_{21} & 0 & 0 & a_{22} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} \end{array}\right) \Leftarrow 6 \times 6$$

In a general case of $2\times 2$ matrix $A$ & $3\times 3$ matrix $B$

$(N_1=2)$ $(N_2=3)$

$$A\otimes B = \left(\begin{array}{ccc|ccc}
a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} \\
a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\
a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} \\
\hline
a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} \\
a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\
a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33}
\end{array}\right)$$

Useful relations

$$\det(A\otimes B) = (\det A)^{N_2}(\det B)^{N_1}$$

($N_1$ - dimensionality of $A$

$N_2$ — — — — of $B$)

$$\mathrm{Tr}(A\otimes B) = (\mathrm{Tr}\,A)(\mathrm{Tr}\,B)$$