Lecture # 23

Applications of time-independent perturbation theory: hyperfine structure of the hydrogen atom.

Consider (H)-atom: $H_0 = \frac{\vec{p}^2}{2\underset{\overset{\shortparallel}{m_e}}{\mu}} - \frac{e^2}{r} \Rightarrow E^{(0)}_{(n)} = -\frac{E_I}{n^2}$

Perturbation: $V = \underbrace{V_{fine}}_{} + \underbrace{V_{hf}}_{} \leftarrow$ hyperfine

V_{fine}: see Lecture #22, p. 1-4
($m(v)$-dependence, $\vec{L}\cdot\vec{S}$ coupling, nonlocality of interaction $\sim \Delta V_{coulomb}(\vec{r})$) ← Darwin term

Energy corrections due to V_{fine}

$\Downarrow$

$$\frac{\Delta E_{fine}}{E^{(0)}} \sim \alpha^2 \sim 10^{-5}$$

$\alpha = \frac{e^2}{\hbar c} \leftarrow$ fine-structure constant

$\Uparrow$ This was all about the electron and its motion

$\Downarrow$

What about hyperfine structure? $\Longrightarrow$ now consider the proton (mass M_p, charge $q \overset{\text{Gauss (units)}}{=} -e = |e|$ or in SI units $q = -\frac{e}{\sqrt{4\pi\varepsilon_0}}$), which is also a spin-$\frac{1}{2}$

particle and has a magnetic moment $\vec{M}_I$ associated with its spin $\vec{I}$:

$$\vec{M}_I = g_p \frac{\mu_{B(n)}}{\hbar} \vec{I}$$

gyromagnetic ratio $g_p \simeq 5.585$

nuclear Bohr magneton $\mu_{B(n)} = \frac{e\hbar}{2m_p c}$

(recall $\mu_B = \frac{e\hbar}{2m_e c}$, $m_e << m_p$ $\Downarrow$ $\mu_{B(n)} << \mu_B$)

for electron

$\vec{M}_S = \frac{2\mu_B}{\hbar} \vec{S}$

spin magnetic moment of the electron

interactions due to the proton spin $\vec{I}$ are <u>much</u> weaker than those due to the electron spin $\vec{S}$

in SI units

$$V_{hf} = -\frac{\mu_0}{4\pi} \Big\{ \underbrace{\frac{q}{m_e r^3} \vec{L} \cdot \vec{M}_I}_{①} + \underbrace{\frac{1}{r^3} \Big[3(\vec{M}_S \cdot \vec{n}) \cdot (\vec{M}_I \cdot \vec{n}) - \vec{M}_S \cdot \vec{M}_I \Big]}_{②} + \underbrace{\frac{8\pi}{3} \vec{M}_S \cdot \vec{M}_I \, \delta(\vec{r})}_{③} \Big\} \qquad (24.1)$$

orbital momentum of the electron

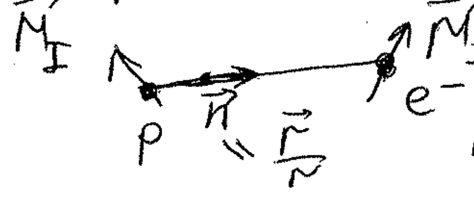
① interaction between $\vec{M}_I$ and magn. field creating by rotating electron

② electron-proton dipole-dipole interaction

interaction of $\vec{M}_s$ with the magn. field created by proton.

③ Fermi's "contact" term

interaction of $\vec{M}_s$ with magnetic field <u>inside</u> the proton



The order of magnitude of the energy correction due to hyperfine interaction $\Rightarrow$

$$\Delta E_{hf} \sim \frac{\mu_0}{4\pi}\cdot\frac{q}{m_e a_0^3}\cdot \underbrace{\hbar}_{\sim|\vec{L}|} \frac{g_p \mu_{B(n)}\cdot \overbrace{\cancel{\hbar}}^{\sim|\vec{I}|}}{\cancel{\hbar}} \sim \frac{\mu_0 q \hbar g_p}{4\pi m_e a_0^3}\cdot$$

$$\cdot\frac{e\hbar}{2m_p c} \sim \frac{\mu_0 q e \hbar^2 g_p}{8\pi c m_e m_p a_0^3} \sim \frac{\mu_0 e^2 \hbar^2 g_p (4\pi\varepsilon_0)}{8\pi m_e m_p \underbrace{a_0^3}_{\frac{\hbar^2}{m_e e^2}} c} =$$

$$\underbrace{=}_{\mu_0\varepsilon_0 = \frac{1}{c^2}} \frac{e^2 \cancel{\hbar^2} g_p m_e^{\cancel{3}\,2} e^6}{2 \cancel{m_e} m_p \hbar^{\cancel{6}\,4} c^2} \sim \frac{e^8 g_p m_e^2}{m_p \hbar^4 c^2}$$

$$\frac{\Delta E_{hf}}{E^{(0)}} \sim \frac{\frac{e^8}{c^2}\frac{g_p m_e^2}{m_p \hbar^4}}{\frac{e^2}{a_0}} \sim \alpha^2 \cdot g_p \underbrace{\frac{m_e}{m_p}}_{\frac{1}{1837}} \Rightarrow$$

much smaller than $\dfrac{\Delta E_{fine}}{E^{(0)}}$!

<u>Example</u> Let's see how V_{hf} affects the energy level of the ground state of the H-atom.

So, 1s - state $\Rightarrow$ 4-fold degenerate (4)
(if we take into account the spins of the electron and the proton)

$\Downarrow$

$$|\underset{n}{1},\underset{l}{0},\underset{m}{0};\underset{m_s}{\pm\tfrac{1}{2}};\underset{m_I}{\pm\tfrac{1}{2}}\rangle$$

$$\Uparrow \qquad \Uparrow$$

$$|m_s| \leq \underset{\frac{1}{2}}{\underset{\shortparallel}{S}} \qquad m_I \leq \underset{\frac{1}{2}}{\underset{\shortparallel}{I}}$$

To find the energy corrections due to V_{hf} $\Rightarrow$ need to compose 4×4 matrix.

First $\Rightarrow$ look for obvious zeros!

(1) in (27.1):

$$\langle 1,0,0;m_s';m_I' \,|\, \overbrace{\vec{L}\cdot\vec{M}_I}^{=L_x M_{I_x} + L_y M_{I_y} + L_z M_{I_z}} \,|\, 1,0,0;m_s,m_I\rangle =$$

m_s' irrelevant, since neither $\vec{L}$ or $\vec{M}_I$ act in spin space of the electron $\rightarrow \mathcal{E}_S$

$= \underline{0} \leftarrow$ for any m_s, m_I

(2) $\langle 1,0,0;m_s',m_I' \,|\, \dfrac{3(\vec{M}_s\cdot\vec{n})(\vec{M}_I\cdot\vec{n}) - \vec{M}_s\cdot\vec{M}_I}{r^3} \,|\, \rightarrow$

$|1,0,0;m_s;m_I\rangle = ?$

5

$$\vec{M}_S = \underbrace{\frac{2\mu_B}{\hbar}}_{\gamma_S} \vec{S} \quad ; \quad \vec{M}_I = \underbrace{g_p \frac{\mu_{B(n)}}{\hbar}}_{\gamma_I} \vec{I}$$

$$\frac{3\gamma_S\gamma_I}{r^3} (\vec{S}\cdot\vec{n})(\vec{I}\cdot\vec{n}) - \underbrace{\frac{\gamma_S\gamma_I}{r^3}}_{= f(r)}\vec{S}\cdot\vec{I} = \Big[3 \underbrace{(\vec{S}\cdot\vec{n})}_{= \sum_i S_i n_i} \underbrace{(\vec{I}\cdot\vec{n})}_{= \sum_j I_j n_j} -$$

$$- \underbrace{\vec{S}\cdot\vec{I}}_{\sum_k S_k I_k}\Big] f(r) =$$

$$= f(r)\Big[3\sum_{i,j} S_i I_j\, n_i n_j - \overbrace{\vec{S}\cdot\vec{I}}^{\sum_{i} \delta_{ij} S_i I_j}\Big] =$$

$$= f(r) \sum_{i,j} S_i I_j \, (3 n_i n_j - \delta_{ij}) = f(r)\cdot\{\text{various}$$

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

Ex. : $i = j = z \Rightarrow \underbrace{3\cos^2\theta - 1}_{\sim Y_2^0} \leftarrow 3 n_i n_j - \delta_{ij}$

$\Rightarrow$ combinations of $S_i I_j \cdot Y_2^q\}$ $\Leftarrow$ show for HW!

$(q = 0, \pm 1, \pm 2)$

Then, $\langle 1,0,0; m_S', m_I' | f(r) \underbrace{\{S_i I_j\}}_{\text{Combinat.}} Y_2^q | 1,0,0; m_S, m_I \rangle$

$$\sim \underbrace{\langle 1,0,0 | f(r) Y_2^q | 1,0,0 \rangle}_{\Downarrow} \cdot \langle m_s', m_I' | \{S_i I_j\} | m_s, m_I \rangle$$

alternatively: $Y_0^0 = \frac{1}{\sqrt{4\pi}}$

$$\int Y_0^{0*} Y_2^q Y_0^0 \, d\Omega = 0 = \frac{1}{\sqrt{4\pi}} \int Y_2^q Y_0^0 \, d\Omega = 0$$

orthogon.

$(q = 0, \pm 1, \pm 2)$

recall selection rules:

$\langle 1,0,0 | Y_2^q | 1,0,0 \rangle$

j' m' k j m

$m' = q + m = q$

$2 \le j' \le 2 \Rightarrow j' = 2$

the only non-zero co…

So, the contribution of ② to 1s hyperfine structure is zero

③ contact term

$$\langle 1,0,0; m_s', m_I' | -\frac{2\mu_0}{3} \gamma_s \gamma_I \vec{S} \cdot \vec{I} \, \delta(\vec{r}) | 1,0,0; m_s, m_I \rangle$$

$$= -\frac{2\mu_0 \gamma_s \gamma_I}{3} \underbrace{\langle 1,0,0 | \delta(\vec{r}) | 1,0,0 \rangle}_{\parallel} \langle m_s', m_I' | \vec{S} \cdot \vec{I} | m_s, m_I \rangle$$

$$\int |\psi_{1s}|^2 \delta(\vec{r}) \, dV = |\psi_{1s}(0)|^2$$

$$= A \langle m_s', m_I' | \vec{I} \cdot \vec{S} | m_s, m_I \rangle$$

const

Recall HW #14 ! $\Rightarrow$ change the basis to $|S = \frac{1}{2}, I = \frac{1}{2}; F, m_F \rangle$,

$\vec{F} = \vec{S} + \vec{I}$ – total angular momentum

$$A\vec{I}\cdot\vec{S} = \frac{A}{2}\left(\vec{F}^2 - \vec{I}^2 - \vec{S}^2\right)$$

$$A\vec{I}\cdot\vec{S}\,|F, m_F\rangle = \frac{A\hbar^2}{2}\left[F(F+1) - I(I+1) - S(S+1)\right]\cdot$$

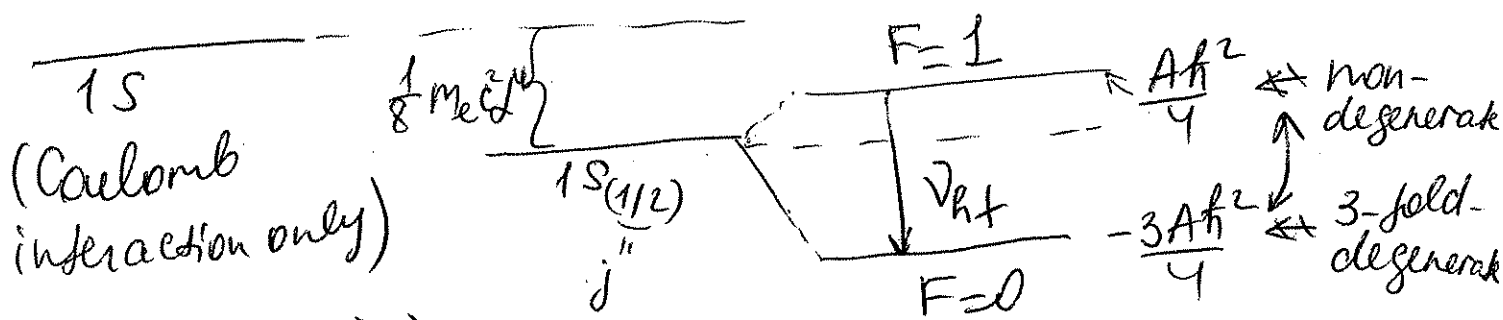
$|F, m_F\rangle$; Since $S = \frac{1}{2}$, $I = \frac{1}{2}$ $\Rightarrow$ possible values for F are 1 and 0

$\Downarrow$

The energy correction due to V_{hf} for 1S level is:

$F = 0 \Rightarrow -\frac{3}{4}A\hbar^2$

$F = 1 \Rightarrow \frac{A\hbar^2}{4}$ $\Longrightarrow$ the degeneracy is partially removed



$\Longrightarrow V_{fine}$ $\quad A \sim \frac{m_e}{m_p}\alpha^4 m_e c^2$

$\Longrightarrow V_{hf}$

$\nu_{hf} = 1\,420\,405\,751.768 \pm 0.001$ Hz $\leftarrow$ (~ 1.4 GHz)

$\uparrow$ measured using the hydrogen maser (1963)

$\Downarrow$ $\lambda_{hf} = 21$ cm $\leftarrow$ detect this emission from interstellar hydrogen clouds (radioastronomy)

Next orders in approximation $\Rightarrow$

Recall from the relativistic Dirac theory (Lecture #26)

$\Downarrow$

$n=2$ — $2s, 2p$ (Coulomb only) $\Rightarrow$ V_{fine} — $2S_{1/2}, 2P_{1/2}$ have the same energy; $2P_{3/2}$

$\Downarrow$

Quantum electrodynamics (QED) $\Longrightarrow$ include interactions between $\vec{M}_S$, $\vec{M}_I$ and quantized electromagnetic field

fluctuat. nature of E&M field

$\Downarrow$

$\nu = 1\,060$ MHz $\rightarrow$ $2S_{1/2}$ — $2P_{1/2}$ $\leftarrow$ Lamb shift

~ 1.06 GHz (same order as ν_{hf} of 1s)

Next level: take into account the structure of the proton!!

$\Downarrow$

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