What if we have more than two angular momenta to add? $\Rightarrow$

If we have $\vec{J}_1, \vec{J}_2, \vec{J}_3 \Rightarrow \vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3 = \underbrace{\vec{J}_{12}}_{\vec{J}_1 + \vec{J}_2} + \vec{J}_3$

$(2j_1+1)(2j_2+1)(2j_3+1)$ - Dimens. space

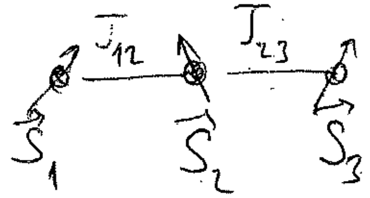
first add j_1 & j_2 and then add j_3

The basis is $|j_1, j_2, j_3; m_1, m_2, m_3\rangle$

In a general case $\Rightarrow \vec{J}_1 + \vec{J}_2 + \dots + \vec{J}_n = \vec{J} \Rightarrow$ first couple $\vec{J}_1 + \vec{J}_2 = \vec{J}_{12}$, then the problem is reduced to the addition of $(n-1)$ angular momenta, etc.

Example (comp exam)

(6) Consider a linear chain of S_1, S_2, S_3 (three identical spins)



$$H = -2J_{12}\,\vec{S}_1 \cdot \vec{S}_2 - 2J_{23}\,\vec{S}_2 \cdot \vec{S}_3, \quad J_{12} = J_{23}$$

Heisenberg exchange interaction

$\vec{S}_{Total} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 \quad ; \quad \vec{S}_{13} = \vec{S}_1 + \vec{S}_3$

- Are S_T and S_{13} good quantum numbers?

Is $\{H, \vec{S}_T^2, \vec{S}_{13}^2, \vec{S}_1^2, \vec{S}_2^2, \vec{S}_3^2, S_{T(z)}\}$ a good set of C.S.C.O?

Check: $[H, \vec{S}_T^2] \Rightarrow [\vec{S}_1 \cdot \vec{S}_2, (\vec{S}_1 + \vec{S}_2 + S_3)^2] = ?$

$$[\vec{S}_2 \cdot \vec{S}_3, (\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2] = ?$$

$$(\vec S_1+\vec S_2+\vec S_3)^2 = \underbrace{(\vec S_1+\vec S_3)^2}_{\vec S_{13}^2} + \vec S_2^{\,2} + 2\vec S_2\cdot\vec S_3 + 2\vec S_2\cdot\vec S_1$$

$$[\vec S_1\cdot\vec S_2,\ (\vec S_1+\vec S_3)^2 + 2\vec S_2\cdot\vec S_3 + 2\vec S_2\cdot\vec S_1 + \vec S_2^{\,2}] =$$

$$= [\vec S_1\cdot\vec S_2,\ 2\vec S_1\cdot\vec S_3 + 2\vec S_2\cdot\vec S_3] = 2\,[S_{1x}S_{2x}+S_{1y}S_{2y}+S_{1z}S_{2z},$$

$$S_{1x}S_{3x}+S_{1y}S_{3y}+S_{1z}S_{3z}] + 2\,[S_{1x}S_{2x}+S_{1y}S_{2y}+S_{1z}S_{2z},\ S_{2x}S_{3x}+$$

$$+S_{2y}S_{3y}+S_{2z}S_{3z}] = 2\,[S_{1x}S_{2x},\ S_{1y}S_{3y}+S_{1z}S_{3z}] + \dots =$$

$$= 2i\hbar\,(S_{1z}S_{2x}S_{3y} - S_{1y}S_{2x}S_{3z} + \dots)$$ ← finish!

If $J_{12}=J_{23}=J$ ← coupling constant (not to be mixed up with ang. momentum)

$$H = -2J(\vec S_1\cdot\vec S_2+\vec S_2\cdot\vec S_3) = -2J\,\frac{\vec S_T^{\,2}-\vec S_{13}^{\,2}-\vec S_2^{\,2}}{2}$$

$$H\,|S_1,S_2,S_T,S_{13}\rangle = -J\,(\hbar^2 S_T(S_T+1) - S_{13}(S_{13}+1)\,\hbar^2 -$$

$$- S_2(S_2+1)\,\hbar^2)\,|S_1,S_2,S_T,S_{13}\rangle \underset{S_1=S_2=S_3=\frac{3}{2}}{=} \underbrace{J\hbar^2\left(-S_T(S_T+1) + S_{13}(S_{13}+1) + \frac{15}{4}\right)}_{E(S_T,S_{13})}|S_1,S_2,S_T,S_{13}\rangle$$

Possible values for S_{13} and S_T? =>

$$S_{13}:\quad \underset{3/2\ \ 3/2}{S_1+S_3},\ S_1+S_3-1,\ \dots\ |S_1-S_3| \Rightarrow \underline{3,2,1,0}$$

$$S_T: \quad S_{13}+S_2,\ S_{13}+S_2-1,\ \ldots,\ |S_{13}-S_2| =$$

$$= 3+\frac{3}{2},\ \ldots\ 3-\frac{3}{2} = \frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}$$

$$2+\frac{3}{2},\ \ldots,\ 2-\frac{3}{2} = \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$$

$$1+\frac{3}{2},\ \ldots\ \left|1-\frac{3}{2}\right| = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$$

$$0+\frac{3}{2},\ \ldots\ \left|0-\frac{3}{2}\right| = \frac{3}{2}$$

altogether, $S_T = \frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$

Possible states & energies

1) $S_{13} = 0 \Rightarrow S_T = \frac{3}{2}$

$$E = J\hbar^2 \cdot \left(-\frac{3}{2}\cdot\frac{5}{2} + \frac{15}{4}\right) = 0$$

2) $S_{13} = 1 \Rightarrow S_T = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

$$E = J\hbar^2\left(-\frac{1}{2}\cdot\frac{3}{2} - 2 + \frac{15}{4}\right) = J\hbar^2$$

. . . .

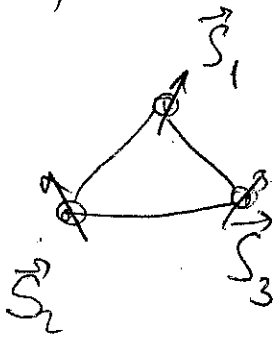
3) $S_{13} = 2 \Rightarrow S_T = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$

. . . .

4) $S_{13} = 3 \Rightarrow S_T = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$

. . . . . .

(c) Now consider



$J_{12} = J_{23} = J$

$J_{13} = J'$

$$H = -2J\underbrace{\left(\vec{S}_1\cdot\vec{S}_2 + \vec{S}_2\cdot\vec{S}_3\right)}_{=\frac{\vec{S}_T^{\,2}-\vec{S}_{13}^{\,2}-\vec{S}_2^{\,2}}{2}} - 2J'\underbrace{\vec{S}_1\cdot\vec{S}_3}_{=\frac{\vec{S}_{13}^{\,2}-\vec{S}_1^{\,2}-\vec{S}_3^{\,2}}{2}}$$

$\Downarrow$

$$E(S_T, S_{13}) = -J\hbar^2\Big(S_T(S_T+1) - S_{13}(S_{13}+1) - \underbrace{S_2}_{=3/2}(S_2+1)\Big) -$$

$$- J'\hbar\Big(S_{13}(S_{13}+1) - \underbrace{S_1}_{=3/2}(S_1+1) - \underbrace{S_3}_{=3/2}(S_3+1)\Big) = +\hbar^2\Big[-S_T(S_T+1)J +$$

$$+ S_{13}(S_{13}+1)(J - J') + \frac{15}{4}J + \frac{15}{2}J'\Big]$$

if $J = J' \Rightarrow E(S_T, S_{13}) = \hbar^2\left[-S_T(S_T+1) + \frac{3}{4}\cdot 15\right]J$

doesn't depend on S_{13} anymore