QM II
Phys 652

Lecture # 14

General formalism for addition of angular momenta

Consider a system with $\vec{J}_1, \vec{J}_2$ $\Rightarrow$

$$\{\vec{J}_1^{\,2}, \vec{J}_2^{\,2}, J_{1z}, J_{2z}\} \qquad \{\vec{J}_1^{\,2}, \vec{J}_2^{\,2}, \vec{J}^{\,2}, J_z\}$$

$\vec{J} = \vec{J}_1 + \vec{J}_2$ ← total angular momentum

$$\{|j_1, j_2, m_1, m_2\rangle\} \xLeftrightarrow{?} \{|j_1, j_2, j, m_j\rangle\}$$

1) How are j, m_j related to j_1, j_2, m_1, m_2
2) What is the connection between the basis $\{|j_1, j_2, m_1, m_2\rangle\}$ and $\{|j_1, j_2, j, m_j\rangle\}$?

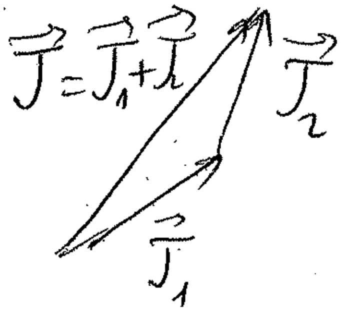
Last week we answered the same question for a simple case of $j_1 = S_1 = \frac{1}{2}$, $j_2 = S_2 = \frac{1}{2}$ $\Rightarrow$

$$? = S = S_1 + S_2 = 1, \quad m_s = -1, 0, 1$$
$$S_1 - S_2 = 0$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle); \qquad |1,1\rangle = |++\rangle$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$
$$|1,-1\rangle = |--\rangle$$

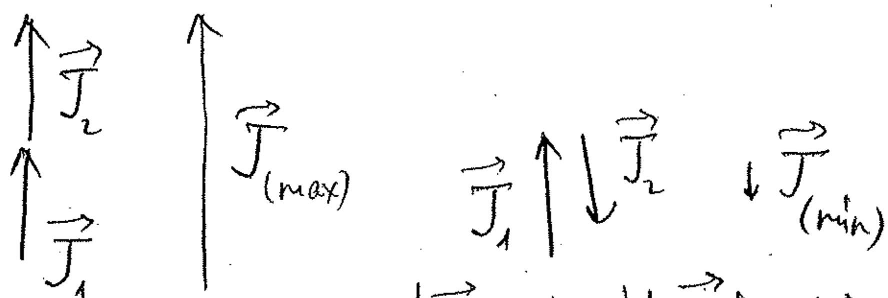
Today we consider general case of $\vec{J}_1, \vec{J}_2$

$$|\vec{J}_{max}| = |\vec{J}_1| + |\vec{J}_2|$$



vector addition

$\Rightarrow$



$$|\vec{J}_{min}| = \big|\, |\vec{J}_1| - |\vec{J}_2| \,\big|$$

Classical mechanics: $|\vec{J}|$ can be anything from $\big|\,|\vec{J}_1| - |\vec{J}_2|\,\big|$ to $|\vec{J}_1| + |\vec{J}_2|$

QM: angular momentum is quantized $\Rightarrow$
p. 456–457 of Sakurai

$$j = \{j_1 + j_2,\ j_1 + j_2 - 1,\ j_1 + j_2 - 2,\ \dots,\ |j_1 - j_2|\}$$

$m_j = m_1 + m_2$, $m_j \le j$ $\Rightarrow$ we have $2j+1$ - states for each j

To find an unambiguous relationship between $\{|j_1, j_2, m_1, m_2\rangle\}$ and $\{|j_1, j_2, j, m_j\rangle\}$

need to ensure that these have the same dimensionality!

Consider $|j_1, j_2; m_1, m_2\rangle = \underset{\substack{\uparrow \\ (2j_1+1)\text{-states}}}{|j_1, m_1\rangle} \otimes \underset{\substack{\uparrow \\ (2j_2+1)\text{ state}}}{\underbrace{|j_2, m_2\rangle}}$

$\mathcal{E}(j_1, j_2) = \mathcal{E}(j_1) \otimes \mathcal{E}(j_2)$

$N_{j_1 j_2} = \underline{(2j_1+1)\cdot(2j_2+1)} \leftarrow$ see Supplementary note

↑ dimensionality of space $\mathcal{E}(j_1, j_2)$

What about $|j_1, j_2; j, m_j\rangle$? $\Rightarrow$ for each j have $2j+1$ states

⇓

$$N_{j_1 j_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) \underset{\substack{\uparrow \\ \text{show!}}}{=} (2j_1+1)(2j_2+1)$$

j varies from $|j_1-j_2|$ to j_1+j_2

⇓

same dimensionality ! $\Rightarrow$

present $\leftarrow$ coupled representation

$$|j_1, j_2; j, m_j\rangle = \underbrace{\sum_{m_1, m_2} |j_1, j_2; m_1, m_2\rangle\langle j_1, j_2, m_1, m_2|}_{I} \tag{16.1}$$

$$|j_1, j_2; j, m_j\rangle = \sum_{m_1 m_2} \underbrace{C(j_1, j_2, j; m_1, m_2, m)}_{\substack{\uparrow \\ \text{Clebsch - Gordan coefficient}}} \underbrace{|j_1, j_2, m_1, m_2\rangle}_{\substack{\uparrow \\ \text{uncoupled}}}$$

also called vector addition coefficients $\leftarrow$ Clebsch - Gordan coefficient

$$C(j_1, j_2, j; m_1, m_2, m_j) = \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m_j \rangle$$

C.-G. coefficients are coefficients of the transformation between the $|j_1, j_2; m_1, m_2\rangle$ and $|j_1, j_2; j, m_j\rangle$ bases

Sometimes C.-G. coeff. are written as ($m_j \equiv m$)

$$\langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle = (-1)^{j_1 - j_2 + m} \cdot \sqrt{2j+1} \cdot$$

$$\cdot \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$

Wigner's 3-j symbol

Properties of C.-G. coefficients

1) $\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \neq 0$ only if $m = m_1 + m_2$, $|j_1 - j_2| \leq j \leq j_1 + j_2$

2) Coefficients are real (convention)

3) $\langle j_1, j_2; j_1, j - j_1 | j_1, j_2; j, j \rangle > 0$ (convention)

4) Orthogonality: $\sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_{m=-j}^{j}\langle m_1,m_2|j,m\rangle\langle m_1',m_2'|j,m\rangle = \delta_{m_1m_1'}\delta_{m_2m_2'}$;

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\langle m_1,m_2|j,m\rangle\langle m_1,m_2|j',m'\rangle = \delta_{jj'}\delta_{mm'}$$

Call $|j_1,j_2;m_1,m_2\rangle \equiv |m_1,m_2\rangle$

$|j_1,j_2;j,m\rangle \equiv |j,m\rangle$

5) $\langle m_1,m_2|j,m\rangle = (-1)^{j_1+j_2-j}\langle -m_1,-m_2|j,-m\rangle$

6) <u>Recursion relations</u>

Let's re-write Eq. (16.1) in terms of simplified notations

$$|j,m\rangle = \sum_{m_1,m_2}\langle m_1,m_2|j,m\rangle\,|m_1,m_2\rangle$$

Act on $|j,m\rangle$ with $J_\pm$:

$$J_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle$$

Act on $|m_1,m_2\rangle$ with $J_{1\pm}$ and $J_{2\pm}$ $\Rightarrow$

$$J_{\substack{1\pm\\2}}|m_1,m_2\rangle = \hbar\sqrt{j_{\substack{1\\2}}(j_{\substack{1\\2}}+1)-m_{\substack{1\\2}}(m_{\substack{1\\2}}\pm1)}\,\substack{|m_1\pm1,m_2\rangle\\|m_1,m_2\pm1\rangle}$$

Since $J_\pm = J_{1\pm}+J_{2\pm}$ $\Rightarrow$

$$J_\pm|j,m\rangle = \sum_{m_1,m_2}\langle m_1,m_2|j,m\rangle\,(J_{1\pm}+J_{2\pm})\,|m_1,m_2\rangle$$

Then,

$$\sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle = \sum_{m_1,m_2}\Big(\langle m_1,m_2|j,m\rangle \cdot \sqrt{j_1(j_1+1)-m_1(m_1\pm1)}\,|m_1\pm1,m_2\rangle + \sqrt{j_2(j_2+1)-m_2(m_2\pm1)} \cdot |m_1,m_2\pm1\rangle\Big);$$

Multiply by $\langle m_1',m_2'|$ $\Rightarrow$ $\quad \delta_{m_2',m_2}\,\delta_{m_1',m_1\pm1}$

$$\langle m_1',m_2'|j,m\pm1\rangle = \sum_{m_1,m_2}\Big[\langle m_1,m_2|j,m\rangle \cdot \underbrace{\langle m_1',m_2'|m_1\pm1,m_2\rangle}_{\delta_{m_2',m_2}\,\delta_{m_1',m_1\pm1}} \cdot \frac{\sqrt{j_1(j_1+1)-m_1(m_1\pm1)}}{\sqrt{j(j+1)-m(m\pm1)}} + \langle m_1,m_2|j,m\rangle \cdot \underbrace{\langle m_1',m_2'|m_1,m_2\pm1\rangle}_{\delta_{m_1',m_1}\,\delta_{m_2',m_2\pm1}} \cdot \frac{\sqrt{j_2(j_2+1)-m_2(m_2\pm1)}}{\sqrt{j(j+1)-m(m\pm1)}}\Big] = \langle m_1'\mp1, m_2'|j,m\rangle \cdot$$

$m_1 = m_1'\mp1$
$m_2 = m_2'$
in 1st term
and
$m_1 = m_1'$
$m_2 = m_2'\mp1$
in the
2nd term

7

$$\cdot \ \frac{\sqrt{j_1(j_1+1)-(m_1'\mp 1)\underbrace{(m_1'\mp 1\pm 1)}_{= m_1'}}}{\sqrt{j(j+1)-m(m\pm 1)}} + \langle m_1', m_2'\mp 1 | j, m\rangle$$

$$\cdot \ \frac{\sqrt{j_2(j_2+1)-m_2'(m_2'\mp 1)}}{\sqrt{j(j+1)-m(m\pm 1)}} \quad \Rightarrow$$

this is a recursion relationship between

$\langle m_1, m_2 | j, m\pm 1\rangle$ and $\langle m_1 \mp 1, m_2 | j, m\rangle$
and $\langle m_1, m_2 \mp 1, j, m\rangle$

7) More relations between C.-G. coefficients

$|j, m\rangle \Rightarrow$ Consider $j = j_1 + j_2$
$m = j = j_1 + j_2$

Convention: $|\underbrace{j_1+j_2}_{j}, \underbrace{j_1+j_2}_{m}\rangle = |j_1, j_2, \underbrace{j_1}_{m_1}, \underbrace{j_2}_{m_2}\rangle$

coupled representation

uncoupled representation

Act with $\underbrace{J_-}|j, m\rangle \Rightarrow$

$J_- |j, m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\, |j, m-1\rangle =$

$$\underset{\uparrow}{=} \hbar \sqrt{\underbrace{(j_1+j_2)(j_1+j_2+1)-(j_1+j_2)(j_1+j_2-1)}} \cdot |\underbrace{j_1+j_2}_{j}, \underbrace{j_1+j_2-1}_{m}\rangle$$

$m = j_1+j_2$

$j = j_1+j_2$

$$\| \quad (j_1+j_2)(\cancel{j_1}+\cancel{j_2}-\cancel{j_1}-\cancel{j_2}+1+1) =$$

$$= (j_1+j_2)\cdot 2$$

$$= \hbar\sqrt{2}\cdot\sqrt{j_1+j_2}\ \underbrace{|j_1+j_2, j_1+j_2-1\rangle} \Rightarrow$$

$$\underbrace{|j_1+j_2, j_1+j_2-1\rangle} = \frac{J_- |j_1+j_2, j_1+j_2\rangle}{\hbar\sqrt{2}\sqrt{j_1+j_2}} =$$

$$\underset{\uparrow}{=} \frac{1}{\hbar\sqrt{2}\sqrt{j_1+j_2}} \left\{ J_{1-}|j_1, j_2, j_1, j_2\rangle + J_{2-}|j_1, j_2, j_1, j_2\rangle \right\}$$

$J_- = J_{1-} + J_{2-}$

convention:

$|j_1+j_2, j_1+j_2\rangle = |j_1, j_2, j_1, j_2\rangle$

$$\underset{\uparrow}{=} \underbrace{\sqrt{\frac{j_1}{j_1+j_2}}\ |j_1, j_2; j_1-1, j_2\rangle} + \underbrace{\sqrt{\frac{j_2}{j_1+j_2}}\ |j_1, j_2; j_1, j_2-1\rangle}$$

show!

<u>Example</u> Calculate $\langle \underbrace{\tfrac{1}{2}}_{j_1}, \underbrace{1}_{j_2}; \underbrace{\tfrac{1}{2}}_{m_1}, \underbrace{0}_{m_2} | \underbrace{\tfrac{3}{2}}_{j}, \underbrace{\tfrac{1}{2}}_{m}\rangle$

Since $|\frac{3}{2}, \frac{1}{2}\rangle \underset{\uparrow}{=} \underbrace{\sqrt{\frac{1/2}{1/2+1}}}_{=\frac{1}{\sqrt{3}}} |\frac{1}{2}, 1; -\frac{1}{2}, 1\rangle + \underbrace{\sqrt{\frac{1}{1/2+1}}}_{=\sqrt{\frac{2}{3}}} |\frac{1}{2}, 1; \frac{1}{2}, 0\rangle$

$j_1+j_2 = \frac{3}{2}$

$j_1+j_2-1 = \frac{1}{2}$

$$\langle \tfrac{1}{2}, 1; \tfrac{1}{2}, 0 | \tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}$$

Useful relations:

$$|\underbrace{j_1+j_2}_{j}, \underbrace{j_1+j_2-1}_{m}\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\, |j_1,j_2; j_1-1, j_2\rangle +$$

$$+ \sqrt{\frac{j_2}{j_1+j_2}}\, |j_1,j_2; j_1, j_2-1\rangle \quad ; \qquad (16.2)$$

$$|j_1+j_2-1, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\, |j_1,j_2; j_1, j_2-1\rangle -$$

$$- \sqrt{\frac{j_2}{j_1+j_2}}\, |j_1,j_2; j_1-1, j_2\rangle \qquad (16.3)$$

More relations:

A. Messiah, "Quantum Mechanics",
vol. 1 & 2 (Bound as one)
pp. 1054 - 1060

Supplementary note

As we showed before, if $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} \Rightarrow S = 0, 1$

Symbolically, it can be presented as $\underbrace{\tfrac{1}{2}}_{\mathcal{E}_{S_1}} \otimes \underbrace{\tfrac{1}{2}}_{\mathcal{E}_{S_2}} = 1 \underset{\substack{\uparrow \\ \text{direct sum}}}{\oplus} 0$

$$\underset{\substack{\uparrow \\ \text{dimensionality of } \mathcal{E}_S}}{N} = N_1 \cdot N_2 = 2 \cdot 2 = 4 = 3 + 1$$

$$\mathcal{E}_S = \mathcal{E}_{S_1} \otimes \mathcal{E}_{S_2} = \underbrace{\mathcal{E}_{S=1}}_{N=3} \oplus \underbrace{\mathcal{E}_{S=0}}_{N=1}$$

Dimensionality of the space check: $(2S_1+1)(2S_2+1) = 4$

Or $\sum_{S=0}^{1} (2S+1) = 1 + 3 = 4$ ✓

Similarly, for any arbitrary j_1 & j_2 $\Rightarrow$

$$\underbrace{j_1 \otimes j_2}_{\text{reducible under rotation in } \mathcal{E}_J} = j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \dots \oplus |j_1 - j_2|$$

irreducible under rotations

$\Downarrow$

see Sakurai p. 215