Problem #1

From the definition of the spherical tensor ⇒

$$\mathcal{D}(R)\, T_q^{(k)}\, \mathcal{D}^{+}(R) = \sum_{q'} \mathcal{D}_{q'q}^{(k)}\, T_{q'}^{(k)}$$

For infinitesimal rotations ⇒

$$\mathcal{D}_{q'q}^{(k)} = \langle k, q' | 1 - \tfrac{i}{\hbar} d\varphi\, \vec{J}\cdot\vec{n} | k, q \rangle$$

Let's find $\mathcal{D}(R)\, T_q^{(k)}\, \mathcal{D}^{+}(R)$:

$$\left(1 - \tfrac{i}{\hbar} d\varphi\, \vec{J}\cdot\vec{n}\right) T_q^{(k)} \left(1 + \tfrac{i}{\hbar} d\varphi\, \vec{J}\cdot\vec{n}\right) = T_q^{(k)} + \frac{i}{\hbar} d\varphi \left[T_q^{(k)}, \vec{J}\cdot\vec{n}\right] + \cancel{O(d\varphi^2)} \qquad (1)$$

On the right-hand side:

$$\sum_{q'} \mathcal{D}_{q'q}^{(k)}\, T_{q'}^{(k)} = \sum_{q'} \langle k, q' | 1 - \tfrac{i}{\hbar} d\varphi\, \vec{J}\cdot\vec{n} | k, q \rangle\, T_{q'}^{(k)}$$

$$= T_q^{(k)} - \frac{i}{\hbar} d\varphi \sum_{q'} \langle k, q' | \vec{J}\cdot\vec{n} | k, q \rangle\, T_{q'}^{(k)} \qquad (2)$$

Problem #3

(a) $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$$xy = r^2\sin^2\theta\,\overbrace{\cos\varphi\sin\varphi}^{\frac{1}{2}\sin 2\varphi}$$
$$xz = r^2\sin\theta\cos\varphi\cos\theta$$
$$x^2-y^2 = r^2\sin^2\theta\,\underbrace{(\cos^2\varphi-\sin^2\varphi)}_{\cos 2\varphi}$$

$\Downarrow$

express in terms of spherical tensors $T_q^{(k)} \leftrightarrow Y_\ell^m$ (i.e. spherical harmonics)

spherical tensor of rank 2

$$T^{(k=2)}_{q=\pm 2} = Y_2^{\pm 2}$$

$$xy = r^2\cdot\frac{1}{4i}\sqrt{\frac{32\pi}{15}}\left(Y_2^2 - Y_2^{-2}\right)$$

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$$\sin^2\theta\sin 2\varphi = \sin^2\theta\,\frac{e^{2i\varphi}-e^{-2i\varphi}}{2i} = \frac{1}{2i}\sqrt{\frac{32\pi}{15}}\left(Y_2^2 - Y_2^{-2}\right)$$

$\sin^2\theta\, e^{2i\varphi} = \sqrt{\frac{32\pi}{15}}Y_2^2$, $\sin^2\theta\, e^{-2i\varphi} = \sqrt{\frac{32\pi}{15}}Y_2^{-2}$

$$xz = r^2\cdot\frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(Y_2^{-1} - Y_2^{1}\right) \leftarrow T^{(k=2)}_{q=\pm 1} = Y_2^{\pm 1}$$

$$Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$\sin\theta\cos\theta\cos\varphi = \sin\theta\cos\theta\,\frac{e^{i\varphi}+e^{-i\varphi}}{2} = \sqrt{\frac{8\pi}{15}}\left(Y_2^{-1} - Y_2^{1}\right)\frac{1}{2}$$

$$x^2-y^2 = r^2 \cdot \frac{1}{2}\sqrt{\frac{32\pi}{15}}\left(Y_2^2+Y_2^{-2}\right)$$

$$\sin^2\theta\cos 2\varphi = \sin^2\theta\,\frac{e^{2i\varphi}+e^{-2i\varphi}}{2} = \sqrt{\frac{32\pi}{15}}\cdot\frac{1}{2}\left(Y_2^2+Y_2^{-2}\right)$$

(b) $Q \equiv e\langle \alpha, j, \underbrace{j}_{m} | 3z^2-r^2 | \alpha, j, \underbrace{j}_{m}\rangle = e r^2 \sqrt{\frac{16\pi}{5}} \cdot \langle \alpha, j, j | Y_2^0 | \alpha, j, j\rangle$

$e\langle \alpha, j, m' | x^2-y^2 | \alpha, j, j\rangle = ?$  $m' = j, j-1, \ldots, -j$

Find possible values for $m'$ $\Rightarrow$ use selection rules

Since $\langle \alpha, j, m' | x^2-y^2 | \alpha, j, j\rangle \overset{(a)}{=} \frac{r^2}{2}\sqrt{\frac{32\pi}{15}} \cdot \langle \alpha, j, m' | Y_2^2+Y_2^{-2} | \alpha, j, j\rangle$,

$m' = j \pm 2 \Rightarrow$ since $|m'| \le j \Rightarrow$ only $m' = j-2$ is acceptable

So, need to calculate

$$e\langle \alpha, j, j-2 | Y_2^{-2} | \alpha, j, j\rangle \cdot \frac{r^2}{2}\sqrt{\frac{32\pi}{15}} \qquad (1)$$

Consider $[J_+, Y_2^{-1}] \underset{\text{HW\#1, Problem 1}}{=} \hbar\sqrt{2\cdot 3-(-1)\cdot 0}\,Y_2^0 = \hbar\sqrt{6}\,Y_2^0$

(**) go to the last page for the expression via C.-G. coeff. as Sakurai wants

$\star\star$ If the goal is only to express 

$$e\langle \alpha j\, j-2 | Y_2^{-2} | \alpha j j\rangle \cdot \frac{r^2}{2}\sqrt{\frac{32\pi}{15}}$$

in terms of $Q$ & C-G coefficients, use the Wigner-Eckart theorem!

$$Q = er^2\sqrt{\frac{16\pi}{5}}\; \underbrace{\langle \alpha j j | Y_2^0 | \alpha j j\rangle}_{\parallel \atop \frac{\langle j 2 j 0 | j j\rangle}{\sqrt{2j+1}}\langle \alpha j \| Y_2 \| \alpha j\rangle}$$

Then, $$\langle \alpha j \| Y_2 \| \alpha j\rangle = \frac{Q\sqrt{2j+1}}{\langle j 2 j 0 | j j\rangle}\,\frac{1}{er^2\sqrt{\frac{16\pi}{5}}}$$

Then, (1) $\Rightarrow$ $$e\frac{r^2}{2}\sqrt{\frac{32\pi}{15}}\; \underbrace{\langle \alpha j\, j-2 | Y_2^{-2} | \alpha j j\rangle}_{\parallel \atop \frac{\langle j 2 j -2 | j\, j-2\rangle}{\sqrt{2j+1}}\langle \alpha j \| Y_2 \| \alpha j\rangle} =$$

$$= \frac{\cancel{er^2}}{2}\sqrt{\frac{32\pi}{15}}\;\frac{\langle j 2 j -2 | j\, j-2\rangle}{\langle j 2 j 0 | j j\rangle}\;\frac{Q}{\cancel{er^2}\sqrt{\frac{16\pi}{5}}} = \frac{Q}{\sqrt{6}}\,\frac{\langle j 2 j -2 | j\, j-2\rangle}{\langle j 2 j 0 | j j\rangle}$$

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Solution of HW #~~5~~ (cont'd) (8)

## Problem #5

$H = -A(\vec{S}_1\cdot\vec{S}_3 + \vec{S}_2\cdot\vec{S}_3)$, $s_1 = s_2 = s_3 = \frac{1}{2}$

Introduce $\vec{S}_{12} = \vec{S}_1 + \vec{S}_2$ and $\vec{S} = \vec{S}_{12} + \vec{S}_3$

Then, $H = -A\underbrace{\vec{S}_{12}\cdot\vec{S}_3}_{\frac{1}{2}[\vec{S}^2 - \vec{S}_{12}^2 - \vec{S}_3^2]} = -\frac{A}{2}[\vec{S}^2 - \vec{S}_{12}^2 - \vec{S}_3^2]$

Since $\hat{H}, \vec{S}^2, \vec{S}_{12}, \vec{S}_3^2$ commute $\Rightarrow$

choose $|s_{12}, s, m\rangle$ basis $\Rightarrow$ $(\equiv |s_1, s_2, s_3; s_{12}, s, m\rangle)$

$\hat{H}|s_{12}, s, m\rangle = -\frac{A}{2}\hbar^2[s(s+1) - s_{12}(s_{12}+1) - s_3(s_3+1)]|s_{12}, s, m\rangle$

$$\underset{s_3=\frac{1}{2}}{=} \underbrace{-\frac{A\hbar^2}{2}\left[s(s+1) - s_{12}(s_{12}+1) - \frac{3}{4}\right]}_{E(s, s_{12})}|s_{12}, s, m\rangle$$

Possible values for $s_{12}$: $s_1+s_2, \ldots s_1-s_2 = 1, 0$

$$\underline{S_{12} = 1} \Rightarrow S = S_{12} + S_3, \ldots |S_{12} - S_3| = $$

$$= \frac{3}{2}, \frac{1}{2}$$

$$\Downarrow$$

$$E\left(\tfrac{3}{2}, 1\right) = -\frac{A\hbar^2}{2}\underbrace{\left[\frac{3}{2}\cdot\frac{5}{2} - 1\cdot 2 - \frac{3}{4}\right]}_{=1} = \underbrace{-\frac{A\hbar^2}{2}}$$

4-fold deg. $\Downarrow$ $|1, \frac{3}{2}, \pm\frac{1}{2}\rangle$, $|1, \frac{3}{2}, \pm\frac{3}{2}\rangle$

$$E\left(\tfrac{1}{2}, 1\right) = -\frac{A\hbar^2}{2}\underbrace{\left[\frac{1}{2}\cdot\frac{3}{2} - 1\cdot 2 - \frac{3}{4}\right]}_{=-2} = \underline{A\hbar^2}$$

two-fold deg. $\Downarrow$ $|\underset{S_{12}}{1}, \underset{S}{\tfrac{1}{2}}, \underset{m}{\pm\tfrac{1}{2}}\rangle$

$$\underline{S_{12} = 0} \Rightarrow S = \frac{1}{2} \Rightarrow$$

$$E\left(\tfrac{1}{2}, 0\right) = -\frac{A\hbar^2}{2}\left[\frac{1}{2}\cdot\frac{3}{2} - \frac{3}{4}\right] = \underbrace{0}$$

two-fold deg.

$\Downarrow$

$$|\underset{S_{12}}{0}, \underset{S}{\tfrac{1}{2}}, \underset{m}{\pm\tfrac{1}{2}}\rangle$$

~~Problem #4~~