Phys 652 QM II

Solution of HW #6

③

Problem #1

(a) $\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \underset{\substack{\uparrow \\ \text{Lecture \#14}}}{=} \frac{\sqrt{2}}{2}(\Psi_+ + \Psi_-) \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{= |s_x = \frac{\hbar}{2}\rangle} + \frac{\sqrt{2}}{2}(\Psi_+ - \Psi_-) \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{= |s_x = -\frac{\hbar}{2}\rangle}$;

$$\mathcal{P}(s_x = +\tfrac{\hbar}{2}) = \int \left(\frac{\sqrt{2}}{2}\right)^2 |\Psi_+ + \Psi_-|^2 dV =$$

$$= \frac{1}{2} \underbrace{\int (|\Psi_+|^2 + |\Psi_-|^2)\, dV}_{= 1 \leftarrow \text{normalization}} +$$

$$+ \frac{1}{2}\int (\Psi_+^* \Psi_- + \Psi_+ \Psi_-^*)\, dV = \frac{1}{2} + \underbrace{\int_0^\infty |R(r)|^2 r^2 dr}_{= 1/2 \leftarrow \text{Lecture \#14}} .$$

$$-\frac{1}{3}\underbrace{\int |Y_1^0|^2 d\Omega}_{=1} \cdot 2 = \frac{1}{2} - \frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{3}$$

Similarly, $\mathcal{P}_{S_x=-\frac{\hbar}{2}} = \int \left(\frac{\sqrt{2}}{2}\right)^2 |(\Psi_+ - \Psi_-)|^2 dV =$

$$= \frac{1}{2}\cdot 1 - \frac{1}{2}\underbrace{\int_0^\infty |R|^2 r^2 dr}_{=1/2} \cdot \frac{-1}{3}\int |Y_1^0|^2 d\Omega \cdot 2 =$$

$$= \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

Check: $\frac{1}{3} + \frac{2}{3} = 1$ ✓

(b) $\langle S_x \rangle = \mathcal{P}_{S_x=+\frac{\hbar}{2}} \cdot \frac{\hbar}{2} + \mathcal{P}_{S_x=-\frac{\hbar}{2}} \cdot \left(-\frac{\hbar}{2}\right) =$

$$= \frac{1}{3}\cdot\frac{\hbar}{2} - \frac{2}{3}\cdot\frac{\hbar}{2} = -\frac{\hbar}{6}$$

(c) Possible outcomes for $L_z \Rightarrow m_{(\ell)} = 0, 1 \Rightarrow$ $L_z = 0$ or $\hbar$

$$\mathcal{P}_{L_z=0} = \underbrace{\int_0^\infty |R(r)|^2 r^2 dr}_{=\frac{1}{2}} \cdot \underbrace{\left[1 + \frac{1}{3} + \frac{1}{3}\right]}_{=\frac{5}{3}} = \frac{5}{6}$$

$\mathcal{P}_{L_z=\hbar} = \int_0^\infty |R(r)|^2 r^2 dr \cdot \frac{1}{3} = \frac{1}{6}$ Check: $\frac{5}{6} + \frac{1}{6} = 1$ ✓

(d) $L^2 = ? \Rightarrow l = 0$ or $1 \Rightarrow$

$L^2 = 0$ or $\hbar^2 l(l+1) \underset{\uparrow}{=} 2\hbar^2$

$l = 1$

$$P_{L^2=0} = \int_0^\infty |R(r)|^2 r^2 dr = \frac{1}{2}$$

$$P_{L^2=2\hbar^2} = \int_0^\infty |R(r)|^2 r^2 dr \cdot \underbrace{\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)}_{=1} = \frac{1}{2}$$

Check: $\frac{1}{2} + \frac{1}{2} = 1$ ✓

(e) $L^2 = 0 \Rightarrow \left[\Psi_{after}\right] = \begin{bmatrix} R(r) Y_0^0 \\ 0 \end{bmatrix} \cdot C$

normalization const

$$C^2 \cdot \underbrace{\int_0^\infty |R(r)|^2 r^2 dr}_{=\frac{1}{2}} \underbrace{\int |Y_0^0|^2 d\Omega}_{=1} = 1 \Rightarrow C = \sqrt{2}$$

$\Downarrow$

$$\left[\Psi_{after}\right] = \begin{bmatrix} \sqrt{2}\, R(r) Y_0^0 \\ 0 \end{bmatrix}$$

(6)

Problem #3

Eq. (10.2):

$$|j_1+j_2, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1,j_2; j_1-1, j_2\rangle +$$

$$+\sqrt{\frac{j_2}{j_1+j_2}}\,|j_1,j_2; j_1, j_2-1\rangle.$$

Present $|j_1+j_2-1, j_1+j_2-1\rangle = \alpha\,|j_1,j_2; j_1-1, j_2\rangle +$

$+\beta\,|j_1,j_2; j_1, j_2-1\rangle.$

Note: since $m = m_1 + m_2$

$$j_1+j_2-1 = (j_1-1) + j_2$$

or

$$j_1 + (j_2-1)$$

can we have other terms? $\Rightarrow$

e.g. $(j_1-2) + (j_2+1) = j_1+j_2-1$?

$\Downarrow$

no! because $m_2 \le |j_2|$

(can't be $m_2 = j_2+1$)

so, only the two terms are in the expansion $\Leftarrow$ same is valid for m_1

$m_1 = j_1-1,\ m_2 = j_2$

$m_1 = j_1,\ m_2 = j_2-1$

From the orthonormality : $|\alpha|^2 + |\beta|^2 = 1$

Since (by convention) the C.-G. coefficients are <u>real</u>

$$\Downarrow$$

$$\underline{\alpha^2 + \beta^2 = 1}$$

Orthonormality :

$$\underbrace{\langle j_1+j_2, j_1+j_2-1|}_{\text{use Eq. (16.2)}} j_1+j_2-1, j_1+j_2-1\rangle = 0$$

$$\sqrt{\frac{j_1}{j_1+j_2}}\,\alpha \underbrace{\langle j_1,j_2; j_1-1, j_2 | j_1,j_2; j_1-1, j_2\rangle}_{=1} +$$

$$+ \sqrt{\frac{j_2}{j_1+j_2}}\,\beta \underbrace{\langle j_1,j_2; j_1, j_2-1 | j_1,j_2; j_1, j_2-1\rangle}_{=1} = 0$$

$$\Downarrow$$

$$\sqrt{j_1}\,\alpha + \sqrt{j_2}\,\beta = 0 \implies \underline{\alpha = -\beta\sqrt{\frac{j_2}{j_1}}}$$

$$\alpha^2+\beta^2 = 1 \implies \beta^2\left(\frac{j_2}{j_1}+1\right) = 1 \implies \beta = \frac{1}{\sqrt{1+\frac{j_2}{j_1}}} = \underline{\sqrt{\frac{j_1}{j_1+j_2}}}$$

$$\alpha = -\beta\sqrt{\frac{j_2}{j_1}} = \underline{-\sqrt{\frac{j_2}{j_1+j_2}}} \implies$$

$$|j_1+j_2-1, j_1+j_2-1\rangle = -\sqrt{\frac{j_2}{j_1+j_2}}\,|j_1,j_2; j_1-1, j_2\rangle + \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1,j_2; j_1, j_2-1\rangle$$

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Problem #2

(a)-(c)

use $|j_1+j_2, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\, |j_1,j_2; j_1-1, j_2\rangle +$

$+ \sqrt{\frac{j_2}{j_1+j_2}}\, |j_1,j_2; j_1, j_2-1\rangle \Rightarrow$

$$|2,1\rangle \underset{\substack{\uparrow \\ j_1=j_2=1}}{=} \frac{1}{\sqrt{2}}|1,1;0,1\rangle + \frac{1}{\sqrt{2}}|1,1;1,0\rangle$$

Act with J_- :

$$J_-|\underbrace{2}_{j},\underbrace{1}_{m}\rangle = \hbar \underbrace{\sqrt{2\cdot 3 - 1\cdot 0}}_{=\sqrt{6}}\, |2,0\rangle$$

From another side,

$$(J_{1-}+J_{2-})|2,1\rangle = (J_{1-}+J_{2-})\frac{1}{\sqrt{2}}\Big(|1,1;0,1\rangle +$$

$$+|1,1;1,0\rangle\Big) = \frac{1}{\sqrt{2}}\Big(J_{1-}|1,1;\underset{m_1}{0},\underset{m_2}{1}\rangle + J_{1-}|1,1;1,0\rangle +$$

$$+ J_{2-}|1,1;0,1\rangle + J_{2-}|1,1;1,0\rangle\Big) = \frac{\hbar}{\sqrt{2}}\Big(\sqrt{1\cdot 2-0}\,|1,1;-1,1\rangle$$

$$+\sqrt{1\cdot 2 - 1\cdot 0}\,|1,1;0,0\rangle + \sqrt{1\cdot 2-1\cdot 0}\,|1,1;0,0\rangle + \sqrt{1\cdot 2 - 0}\,|1,1;1,-1\rangle$$

$=$

$\ominus \hbar \left(|1,1;-1,1\rangle + 2|1,1;0,0\rangle + |1,1;1,-1\rangle \right)$ ②

So, $|2,0\rangle = \frac{\cancel{\hbar}}{\cancel{\hbar}\sqrt{6}} \left(|1,1;-1,1\rangle + 2|1,1;0,0\rangle + \right.$

$\left. + |1,1;1,-1\rangle \right);$

Then,

(a) $\langle 1,1;-1,1|2,0\rangle = \frac{1}{\sqrt{6}}$

(b) $\langle 1,1;1,-1|2,0\rangle = \frac{1}{\sqrt{6}}$

(c) $\langle 1,1;0,0|2,0\rangle = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$

parts (d) and (e)

$|1,0\rangle = ?$

$J_- \underbrace{|1,1\rangle} = \hbar\sqrt{1\cdot 2 - 1\cdot 0}\, |1,0\rangle = \hbar\sqrt{2}\, |1,0\rangle$

use $|j_1+j_2-1, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\, |j_1,j_2;j_1,j_2-1\rangle$

$- \sqrt{\frac{j_2}{j_1+j_2}}\, |j_1,j_2;j_1-1,j_2\rangle$

for $j_1 = j_2 = 1 \Rightarrow j_1+j_2-1 = 1 \Rightarrow$

$|1,1\rangle = \frac{1}{\sqrt{2}}|1,1;1,0\rangle - \frac{1}{\sqrt{2}}|1,1;0,1\rangle$

Then, act on $|1,1\rangle$ with J_- :

$$\underbrace{(J_{1-} + J_{2-})}_{= J_-}\left[\frac{1}{\sqrt{2}}|1,1;1,0\rangle - \frac{1}{\sqrt{2}}|1,1;0,1\rangle\right] =$$

$$= \frac{1}{\sqrt{2}}\Big(J_{1-}|\overset{j_1}{1},\overset{j_2}{1};\overset{m_1}{1},\overset{m_2}{0}\rangle - J_{1-}|1,1;0,1\rangle + J_{2-}|1,1;1,0\rangle$$

$$- J_{2-}|1,1;0,1\rangle\Big) = \frac{\hbar}{\sqrt{2}}\Big(\cancel{\sqrt{1\cdot 2 - 1\cdot 0}\,|1,1;0,0\rangle} -$$

$$- \sqrt{1\cdot 2 - 0\cdot(-1)}\,|1,1;-1,1\rangle + \sqrt{1\cdot 2 - 0}\,|1,1;1,-1\rangle -$$

$$\cancel{- \sqrt{1\cdot 2 - 1\cdot 0}\,|1,1;0,0\rangle}\Big) = \frac{\hbar\sqrt{2}}{\sqrt{2}}\Big(|1,1;1,-1\rangle - |1,1;-1,1\rangle\Big)$$

So, $\cancel{\hbar}\sqrt{2}\,|1,0\rangle = \cancel{\hbar}\Big(|1,1;1,-1\rangle - |1,1;-1,1\rangle\Big)$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\Big(|1,1;1,-1\rangle - |1,1;-1,1\rangle\Big)$$

(d) $\langle 1,1;-1,1|1,0\rangle = -\frac{1}{\sqrt{2}}$

(e) $\langle 1,1;1,-1|1,0\rangle = \frac{1}{\sqrt{2}}$

parts (f) & (g)

$$|\underbrace{0}_{j},\underbrace{0}_{m}\rangle = ?$$ ← in terms of the uncoupled basis

Need $\langle \overset{j_1}{1}, \overset{j_2}{1}; \overset{m_1}{-1}, \overset{m_2}{1} | \overset{j}{0}, \overset{m}{0}\rangle$ & $\langle 1,1;0,0|0,0\rangle$

For $j_1 = j_2 = 1$ → what are possible terms in the expansion of $|0,0\rangle$?

⇓

$$\underbrace{m}_{=0} = m_1 + m_2 \Rightarrow m_1 = -m_2$$

Since $m_1; m_2 = -1, 0, 1$

⇓

can have $|1,1;1,-1\rangle$

$|1,1;-1,1\rangle$

⇓ $|1,1;0,0\rangle$

$$|0,0\rangle = \alpha|1,1;-1,1\rangle + \beta|1,1;0,0\rangle +$$

$$+\gamma|1,1;1,-1\rangle$$, where α, β, γ are real numbers

Normalization: $\langle 0,0|0,0\rangle = 1 \Rightarrow \alpha^2+\beta^2+\gamma^2 = 1$

Orthogonality: $\langle 1,0|0,0\rangle = \langle 2,0|0,0\rangle = 0$

From (a) $\Rightarrow \langle 2,0| = \frac{1}{\sqrt{6}}\big(\langle 1,1;-1,1| +$

$+ 2\langle 1,1;0,0| + \langle 1,1;1,-1|\big)$

From (d) $\Rightarrow$ $\langle 1,0| = \frac{1}{\sqrt{2}}\Big(\langle 1,1;1,-1| -$

$- \langle 1,1;-1,1|\Big)$

Then, $\langle 1,0|0,0\rangle = 0 = -\alpha \cdot \frac{1}{\sqrt{2}} + \frac{\gamma}{\sqrt{2}} \Rightarrow$

$\boxed{\alpha = \gamma}$

$\langle 2,0|0,0\rangle = 0 = \frac{\alpha}{\sqrt{6}} + \frac{2\beta}{\sqrt{6}} + \frac{\gamma}{\sqrt{6}} \Rightarrow$

$\alpha + \gamma + 2\beta = 0 \Rightarrow 2(\alpha+\beta) = 0 \Rightarrow \boxed{\alpha = -\beta}$

Since $\alpha^2 + \beta^2 + \gamma^2 = 1 \Rightarrow \alpha^2 + \alpha^2 + \alpha^2 = 1 \Rightarrow$

$\underline{\alpha = \frac{1}{\sqrt{3}} = \gamma = -\beta}$

Then,

(f) $\langle 1,1;-1,1|0,0\rangle = \alpha = \frac{1}{\sqrt{3}}$

(g) $\langle 1,1;0,0|0,0\rangle = \beta = -\frac{1}{\sqrt{3}}$

Problem #4

$^2P_{3/2}$ state: $\ell = 1$
$s = 1/2$
$j = 3/2$

Also known that $m = -\frac{1}{2}$

coupled

Need to present the state $|j = \frac{3}{2}, m = -\frac{1}{2}\rangle$
in terms of the uncoupled states $\{|\underset{=\ell}{1}, \underset{=s}{\tfrac{1}{2}}; m_\ell, m_s\rangle\}$

We know that

$$\Big|\underbrace{\tfrac{3}{2}}_{\ell+s}, \underbrace{\tfrac{1}{2}}_{\ell+s-1}\Big\rangle = \sqrt{\frac{1}{1+\frac{1}{2}}}\,\Big|1, \tfrac{1}{2}; 0, \tfrac{1}{2}\Big\rangle + \sqrt{\frac{1/2}{3/2}}\,\Big|1, \tfrac{1}{2}; 1, -\tfrac{1}{2}\Big\rangle$$

$$= \sqrt{\frac{2}{3}}\,\Big|1, \tfrac{1}{2}; 0, \tfrac{1}{2}\Big\rangle + \frac{1}{\sqrt{3}}\Big|1, \tfrac{1}{2}; 1, -\tfrac{1}{2}\Big\rangle$$

Let's apply J_- on the left-hand-side:

$$J_-\Big|\tfrac{3}{2}, \tfrac{1}{2}\Big\rangle = \hbar\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{1}{2}\cdot\left(-\tfrac{1}{2}\right)}\,\Big|\tfrac{3}{2}, -\tfrac{1}{2}\Big\rangle =$$

$$= 2\hbar\,\Big|\tfrac{3}{2}, -\tfrac{1}{2}\Big\rangle$$

Now apply $J_- = J_{1-} + J_{2-}$ on the right-hand side:

$$\Big(\overbrace{J_{1-}}^{=L_-} + \overbrace{J_{2-}}^{=S_-}\Big)\left(\sqrt{\tfrac{2}{3}}\,\Big|1,\tfrac{1}{2};\overset{m_\ell}{0},\overset{m_s}{\tfrac{1}{2}}\Big\rangle + \frac{1}{\sqrt{3}}\Big|1,\tfrac{1}{2};1,-\tfrac{1}{2}\Big\rangle\right)$$

$$= \sqrt{\tfrac{2}{3}}\,\hbar\left(\underbrace{\sqrt{1\cdot 2 - 0\cdot(-1)}}_{=\sqrt{2}}\Big|1,\tfrac{1}{2};-1,\tfrac{1}{2}\Big\rangle + \underbrace{\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\cdot\left(-\tfrac{1}{2}\right)}}_{=1}\Big|1,\tfrac{1}{2};0,-\tfrac{1}{2}\Big\rangle\right) + \frac{1}{\sqrt{3}}\hbar\left(\underbrace{\sqrt{1\cdot 2 - 1\cdot 0}}_{=\sqrt{2}}\Big|1,\tfrac{1}{2};0,-\tfrac{1}{2}\Big\rangle + 0\right)$$

$$= \hbar\sqrt{\tfrac{2}{3}}\cdot 2\,\Big|1,\tfrac{1}{2};0,-\tfrac{1}{2}\Big\rangle + \frac{2}{\sqrt{3}}\hbar\,\Big|1,\tfrac{1}{2};-1,\tfrac{1}{2}\Big\rangle \underset{\substack{\uparrow\\ \text{from}\\ \text{p. 1}}}{=} 2\hbar\,\Big|\tfrac{3}{2},-\tfrac{1}{2}\Big\rangle \Rightarrow$$

$$\Big|\tfrac{3}{2},-\tfrac{1}{2}\Big\rangle = \sqrt{\tfrac{2}{3}}\,\Big|1,\tfrac{1}{2};0,\underbrace{-\tfrac{1}{2}}_{m_s}\Big\rangle + \frac{1}{\sqrt{3}}\Big|1,\tfrac{1}{2};-1,\underbrace{\tfrac{1}{2}}_{m_s}\Big\rangle$$

The probability to find the electron at the state with $m_s = \frac{1}{2}$ is $\left(\frac{1}{\sqrt{3}}\right)^2 = \underline{\frac{1}{3}}$