Phys 651 QM

Lecture #9

①

Complete sets of commuting observables

Consider an observable A and a basis of its eigenvectors $\{|\varphi_n\rangle\} \in \mathcal{E}$. If none of the eigenvalues a_n is degenerate $\Rightarrow$ all the eigenspace $\mathcal{E}_n$ are one-dimensional, i.e. to each eigenvalue corresponds a different eigenvector. In other words, specifying the eigenvalue determines in a unique way the corresponding eigenvector (to within a constant factor). In this case, the observable A constitutes by itself a complete set of commuting observables, or C.S.C.O

If, on the other hand, one or several eigenvalues of A are degenerate $\Rightarrow$ specifying a_n is no longer sufficient to characterize a basis vector, since there correspond several independent vectors to degenerate eigenvalues.

Recall ~~Lecture #8~~, the case with $L_z^2 = 1 \Rightarrow$
Appendix of Lecture #8
we had to introduce vectors

$|L_z^2 = 1 ; 1 \rangle$, $|L_z^2 = 1 ; 2 \rangle$, since specifying just $|L_z^2 = 1 \rangle$ was not enough.

Obviously, in this case the basis of eigenvectors of A is not unique ! $\Leftrightarrow$ one can choose any basis inside each of the eigenspaces $\mathcal{E}_n$ of dimension greater than 1.

Is there any way to find a unique basis in this case? In other words, how can we better define $|a_n ; 1 \rangle$, $|a_n ; 2 \rangle$, ... ?

vector #1 in a subspace of eigenvectors corresponding to the eigenvalue a_n

$\Downarrow$

Let's choose another observable B which commutes with A and construct an orthonormal basis of eigenvectors common to A and B.
A & B form a C.S.C.O if this basis is unique (to within a phase factor for each of the

basis vectors), i.e. if, to each of the possible pairs of eigenvalues $\{a_n, b_m\}$, there corresponds only <u>one</u> basis vector.

<u>Example</u>

Eigenvalues of an observable A are $a_1 = 0$

$a_{2,3} = 1$

↑ double-degenera

Corresponding eigenvectors:

$$|a_1 = 0\rangle \doteq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|a_2 = 1\rangle \equiv |a = 1; 1\rangle \doteq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|a_3 = 1\rangle \equiv |a = 1; 2\rangle \doteq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\{A\}$ is <u>not</u> a C.S.C.O.

Basis $|a=0\rangle$, $|a=1\rangle$ is not unique $\Rightarrow$ can introd

$|a=0\rangle$, $|a=1;1\rangle$, $|a=1;2\rangle$,

but how do you choose which vector is #1 and which one is #2? $\Rightarrow$

Consider another observable B, such that

$[A, B] = 0$.

Let's say that the eigenvalues of B are $b_1 = 0$

$b_2 = 1$,

and the eigenvectors are

$$|b=0\rangle \doteq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|b=1\rangle \doteq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|b=-1\rangle \doteq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\{B\}$ is a C.S.C.O

Does this set $|b=0\rangle, |b=1\rangle, |b=-1\rangle$ form a unique basis? $\Rightarrow$ yes! since there are no degenerate eigenvalues.

Also it helps specifying which vector is $|a=1;1\rangle$ and which one is $|a=1,2\rangle$ $\Rightarrow$

introduce a notation $|a=\ldots, b=\ldots\rangle \Rightarrow$

then in our example
$\{A,B\}$ is a C.S.C.O

$$|a=0, b=0\rangle \doteq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|a=1, b=1\rangle \doteq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|a=1, b=-1\rangle \doteq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

now we know exactly what eigenvector is #1 or #2

What if the eigenvalues of B are also degenerate?

then find an observable C, $[B,C]=0 \Rightarrow$ construct the basis $|a=\ldots, b=\ldots, c=\ldots\rangle$

More examples

The set of the three operators X, Y, Z constitutes a C.S.C.O in $\mathcal{E}_{\vec{r}}$, since the three eigenvalues x_0, y_0, z_0 of X, Y, Z uniquely determines the corresponding vector $|\vec{r}_0\rangle$.

Similarly, P_x, P_y, P_z constitute a C.S.C.O in $\mathcal{E}_{\vec{r}}$

<u>Note</u>: in $\mathcal{E}_{\vec{r}}$, X does <u>not</u> constitute a C.S.C.O by itself, since each eigenvalue x_0 is infinitely degenerate (because for each x_0 there are infinite possibilities for y_0, z_0)

However, in a 1D problem space, i.e. $\mathcal{E}_x$, the eigenvalue x_0 uniquely determines the corresponding eigenket $|x_0\rangle$ and therefore, constitutes a C.S.C.O.

<u>Also note</u>: sometimes instead of writing out $|a=\ldots, b=\ldots, c=\ldots, \ldots\rangle \Rightarrow$ a <u>collective</u> index such that $|K\rangle = |a, b, c, \ldots\rangle$ is introdu[ced]

Change of basis. Unitary transformation

A unitary tranformation is the application o' a unitary operator $\hat{U}$ to kets, bras, operators (depending on what we want to transform).

State vectors $|\psi\rangle$ and bras $\langle\psi|$ transform as follows:

new basis $\rightarrow$ $|\psi'\rangle = \hat{U}|\psi\rangle$ $\leftarrow$ old basis ; $\langle\psi'| = \langle\psi|\hat{U}^+$

What about operators $\hat{A}$?

$$\hat{A} \to \hat{A}' ? \Rightarrow$$

$$\hat{A}|\psi\rangle = |\varphi\rangle \underset{\text{unitary transformation}}{\Longrightarrow} \hat{A}'\underbrace{|\psi'\rangle}_{\hat{U}|\psi\rangle} = \underbrace{|\varphi'\rangle}_{\hat{U}|\varphi\rangle}$$

$$\hat{A}'\hat{U}|\psi\rangle = \hat{U}|\varphi\rangle$$

Multiply by $\hat{U}^+$ $\Rightarrow$ $\hat{U}^+\hat{A}'\hat{U}|\psi\rangle = |\varphi\rangle$

since $\hat{U}^+\hat{U} = I$ ← identity

$$\underline{\hat{A} = \hat{U}^+\hat{A}'\hat{U}} \Rightarrow$$

$$\hat{A}\hat{U}^+ = \hat{U}^+\hat{A}'\underbrace{\hat{U}\hat{U}^+}_{I}$$

$$\hat{U}\hat{A}\hat{U}^+ = \underbrace{\hat{U}\hat{U}^+}_{}\hat{A}' \Rightarrow \boxed{\hat{A}' = \hat{U}\hat{A}\hat{U}^+}$$

What if we want to express old basis in terms of a new one? $\Longrightarrow$

$$|\psi\rangle = \hat{U}^{+}|\psi'\rangle \; ; \quad \langle\psi| = \langle\psi'|\hat{U},$$

$$\hat{A} = \hat{U}^{+}\hat{A}'\hat{U}$$

Properties of unitary transformations

- If $\hat{A}$ is Hermitian, $\hat{A}'$ is also Hermitian

$$\hat{A}^{+} = (\hat{U}^{+}\hat{A}'\hat{U})^{+} = \hat{U}^{+}\hat{A}'^{+}\hat{U} = \hat{A} =$$

$$= \hat{U}^{+}\hat{A}'\hat{U} \quad \Longrightarrow \hat{A}'^{+} = \hat{A}'$$

- $\hat{A}$ and $\hat{A}'$ have the same eigenvalues $\Longrightarrow$

$$\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle \Longrightarrow \hat{A}'|\varphi_n'\rangle = a_n|\varphi_n'\rangle$$

$$\hat{A}'|\varphi_n'\rangle = (\hat{U}\hat{A}\underbrace{\hat{U}^{+})(\hat{U}}_{=I}|\varphi_n\rangle) = a_n(\hat{U}|\varphi_n\rangle)$$

$$= a_n|\varphi_n'\rangle$$

Important:

Every Hermitian matrix can be diagonalized by a unitary change of basis!

• Commutators that are equal to (complex) numbers remain unchanged ⇒ ⑧

$$[\hat{A}', \hat{B}'] = [\hat{U}\hat{A}\hat{U}^+, \hat{U}\hat{B}\hat{U}^+] = \hat{U}\underbrace{[\hat{A}, \hat{B}]}_{= \underset{\sim}{a}}\hat{U}^+ = \underset{\sim}{a}$$

• If $\hat{A} = \beta\hat{B} + \gamma\hat{C} \Rightarrow \hat{A}' = \beta\hat{B}' + \gamma\hat{C}'$

$\hat{A} = \alpha\hat{B}\hat{C}\hat{D} \Rightarrow \hat{A}' = \alpha\hat{B}'\hat{C}'\hat{D}'$

• $\langle\psi|\hat{A}|\chi\rangle$ does not change

$$\langle\psi'|\hat{A}'|\chi'\rangle = (\langle\psi|\hat{U}^+)(\hat{U}\hat{A}\hat{U}^+)(\hat{U}|\chi\rangle) =$$

$$= \langle\psi|\underbrace{(\hat{U}^+\hat{U})}_{= I}\hat{A}\underbrace{(\hat{U}^+\hat{U})}_{= I}|\chi\rangle = \langle\psi|\hat{A}|\chi\rangle$$

If $\hat{A} = \hat{I}$ ← identity matrix (operator) $\Rightarrow \langle\psi'|\chi'\rangle = \langle\psi|\chi\rangle$

$\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle \Leftarrow$ invariant under unitary transformations

the norm is conserved

• $(\hat{U}\hat{A}\hat{U}^+)^n = \hat{U}\hat{A}^n\hat{U}^+$

• $\hat{U}f(\hat{A})\hat{U}^+ = f(\hat{U}\hat{A}\hat{U}^+) = f(\hat{A}')$