QM
Phys 651

Lecture #8

Compatible and incompatible observables

Two observables are compatible if their corresponding operators commute $\Rightarrow [A,B]=0$
Otherwise $\Rightarrow$ incompatible

Theorem: If two observables are compatible, their corresponding operators possess a set of common eigenstates

Proof: for simplicity $\Rightarrow$ consider a non-degenerate case

$$A|\varphi_n\rangle = a_n|\varphi_n\rangle \quad (a_m \neq a_n \text{ if } n\neq m)$$

$$\langle\varphi_m|[A,B]|\varphi_n\rangle \underset{\substack{\uparrow \\ \text{compatible} \\ \text{observables}}}{=} 0 = \langle\underbrace{\varphi_m|}_{a_m\langle\varphi_m|} AB-BA \underbrace{|\varphi_n\rangle}_{a_n|\varphi_n\rangle} =$$

$$=(a_m-a_n)\langle\varphi_m|B|\varphi_n\rangle \underset{a_m\neq a_n}{\Rightarrow} \langle\varphi_m|B|\varphi_n\rangle = 0 \underset{(m\neq n)}{} \Rightarrow$$

off-diagonal elements of matrix representing B in $\{|\varphi_n\rangle\}$

possible only if $B|\varphi_n\rangle = b_n|\varphi_n\rangle$

$\Downarrow$

$\{|\varphi_n\rangle\}$ are joint (or simultaneous) eigenstates of A & B

Now let's consider consequtive measurements of A & B in the case of $[A,B]=0$ and $\neq 0$ $\Rightarrow$ ②

1) A & B compatible $([A,B]=0)$

$|\psi\rangle \underset{\substack{\text{measure } A, \\ \text{get } a_n}}{\Longrightarrow} |\varphi_n\rangle \underset{\text{measure } B,\ \text{get } b_n}{\Longrightarrow} |\varphi_n\rangle$

initial state

$A|\varphi_n\rangle = a_n|\varphi_n\rangle$ $\quad B|\varphi_n\rangle = b_n|\varphi_n\rangle$

$\mathcal{P}_n = |\langle\psi|\varphi_n\rangle|^2$ $\quad \mathcal{P}_n = 1$

$$\mathcal{P}(a_n, b_n) = |\langle\psi|\varphi_n\rangle|^2$$

reverse order $\Rightarrow$

$|\psi\rangle \underset{\substack{\text{measure } B, \\ \text{get } b_n}}{\Longrightarrow} |\varphi_n\rangle \underset{\substack{\text{measure } A, \\ \text{get } a_n}}{\Longrightarrow} |\varphi_n\rangle$

$$\mathcal{P}(b_n, a_n) = |\langle\psi|\varphi_n\rangle|^2 = \mathcal{P}(a_n, b_n)$$

2) A & B are incompatible $([A,B]\neq 0)$

$|\psi\rangle \underset{\substack{\text{measure } A, \\ \text{get } a_n}}{\Longrightarrow} |\varphi_n\rangle \underset{\substack{\text{measure } B, \\ \text{get } b_n}}{\Longrightarrow} |\chi_n\rangle$

$\mathcal{P}_n = |\langle\psi|\varphi_n\rangle|^2$ $\quad B|\chi_n\rangle = b_n|\chi_n\rangle$, $\mathcal{P} = |\langle\varphi_n|\chi_n\rangle|^2$

$$\mathcal{P}(a_n, b_n) = |\langle\psi|\varphi_n\rangle|^2 \, |\langle\varphi_n|\chi_n\rangle|^2$$

reverse order $\Rightarrow$

$|\psi\rangle \underset{\substack{\text{measure } B \\ \text{get } b_n}}{\Longrightarrow} |\chi_n\rangle \underset{\substack{\text{measure } A, \\ \text{get } a_n}}{\Longrightarrow} |\varphi_n\rangle$

$$\mathcal{P}(b_n, a_n) = |\langle\psi|\chi_n\rangle|^2 \, |\langle\varphi_n|\chi_n\rangle|^2 \neq \mathcal{P}(a_n, b_n) \,!$$

Example of measurements of incompatible observables

Consider $\{|\varphi_n\rangle\}$, which is an eigenbasis of the Hamiltonian $\Rightarrow$ $H|\varphi_n\rangle = n^2 \mathcal{E}_0 |\varphi_n\rangle$

The system is initially in state $|\varphi_3\rangle$.

What values for the energy and the observable A, whose action on $|\varphi_n\rangle$ is defined by

$$A|\varphi_n\rangle = n a_0 |\varphi_{n+1}\rangle$$

will be obtained if we measure:

(i) H, then A

(ii) A, then H

(i) The measurement of H and then A is represented by $A\underbrace{[H|\varphi_3\rangle]}_{\text{(resulting state)}} \Rightarrow$

$$H|\varphi_3\rangle = 3^2 \mathcal{E}_0 |\varphi_3\rangle = \underbrace{9\mathcal{E}_0}_{\text{result for } H} \overbrace{|\varphi_3\rangle}^{\text{state after measurement}}$$

$$A\underbrace{(H|\varphi_3\rangle)}_{\text{(resulting state)}} \Rightarrow A|\varphi_3\rangle = \underbrace{3a_0}_{\text{result for } A} |\varphi_4\rangle$$

(ii) Now ~~A~~ first and then H $\Rightarrow$

$$A|\varphi_3\rangle = \underbrace{3a_0}_{\text{result for } A} |\varphi_4\rangle \Rightarrow H|\varphi_4\rangle = \underbrace{16\mathcal{E}_0}_{\text{result for } H} |\varphi_4\rangle$$

Results of the experiments depend on the order !

$\Downarrow$

how can we understand why?

$\Downarrow$

Consider $[H, A] \, |\varphi_3\rangle = HA|\varphi_3\rangle - AH|\varphi_3\rangle =$

$= H \cdot 3a_0 |\varphi_4\rangle - A \cdot 9\mathcal{E}_0 |\varphi_3\rangle =$

$= (3a_0 \cdot 16\,\mathcal{E}_0 - 9\,\mathcal{E}_0 \cdot 3a_0) |\varphi_4\rangle =$

$= (48\, a_0 \mathcal{E}_0 - 27\, a_0 \mathcal{E}_0) \, |\varphi_4\rangle = \underbrace{21\, \mathcal{E}_0 a_0 |\varphi_4\rangle}_{\neq 0}$

$\Downarrow$

$[H, A] \neq 0$

$\Downarrow$

results of successive measurements of A & H will depend on their order

The Uncertainty Relation

Consider arbitrary operators $\hat{A}$ and $\hat{B}$ and their expectation values $\langle\hat{A}\rangle$, $\langle\hat{B}\rangle$ with respect to a normalized state vector $|\psi\rangle$:

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$
$$\langle\hat{B}\rangle = \langle\psi|\hat{B}|\psi\rangle$$

The uncertainties ΔA and ΔB are defined by

$$\Delta A = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}\ ; \quad \Delta B = \sqrt{\langle\hat{B}^2\rangle - \langle\hat{B}\rangle^2}, \qquad (8.1)$$

where $\langle\hat{A}^2\rangle = \langle\psi|\hat{A}^2|\psi\rangle$, $\langle\hat{B}^2\rangle = \langle\psi|\hat{B}^2|\psi\rangle$

Note: ΔA, ΔB are numbers, not operators, with definition above.
In Sakurai, $\underset{\substack{\uparrow \\ \text{from Eq. (8.1)}}}{\Delta A} \equiv \sqrt{\langle(\Delta A)^2\rangle}$, while ΔA denotes an operator $A - \langle A$

Here, to simplify notations, we will use ΔA, ΔB as defined in Eq. (8.1) (numbers).

Let's also define operators $\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$
$\Delta\hat{B} = \hat{B} - \langle\hat{B}\rangle$ $\Rightarrow$

What's the relationship between $\Delta\hat{A}$ and ΔA?

$\Downarrow$

$$(\Delta\hat{A})^2 = (\hat{A} - \langle\hat{A}\rangle)^2 = \hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle + \langle\hat{A}\rangle^2$$

$\Downarrow$

$$\langle(\Delta\hat{A})^2\rangle = \langle\hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle + \langle\hat{A}\rangle^2\rangle =$$

$$= \langle\hat{A}^2\rangle - 2\langle\hat{A}\rangle^2 + \langle\hat{A}\rangle^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 =$$

$$= (\underbrace{\Delta A}_{\text{uncertainty}})^2$$

Consider $|\chi\rangle = \Delta\hat{A}|\Psi\rangle$
$|\varphi\rangle = \Delta\hat{B}|\Psi\rangle$ $\Rightarrow$ apply Schwarz inequality

$\Downarrow$

$$|\langle\chi|\varphi\rangle|^2 \leq \langle\chi|\chi\rangle\langle\varphi|\varphi\rangle$$

$\Downarrow$

$$\langle\chi|\chi\rangle = \langle\Psi|\Delta\hat{A}^{+}\Delta A|\Psi\rangle = \langle\Psi|(\Delta\hat{A})^2|\Psi\rangle =$$

$$= \langle(\Delta\hat{A})^2\rangle = (\Delta A)^2$$

consider $\hat{A}$ to be Hermitian $\Rightarrow$ $\Delta\hat{A}$ is also Hermitian

Similarly, $\langle\varphi|\varphi\rangle = \langle(\Delta\hat{B})^2\rangle$

$$\langle\chi|\varphi\rangle = \langle\Psi|\Delta\hat{A}\,\Delta\hat{B}|\Psi\rangle = \langle(\Delta\hat{A}\,\Delta\hat{B})\rangle$$

Then, $$|\langle(\Delta\hat{A}\,\Delta\hat{B})\rangle|^2 \leq (\Delta A)^2(\Delta B)^2 \tag{8.2}$$

$$\Delta\hat{A}\,\Delta\hat{B} = \frac{1}{2}\underbrace{[\Delta\hat{A},\Delta\hat{B}]}_{\text{commutador}} + \frac{1}{2}\underbrace{\{\Delta\hat{A},\Delta\hat{B}\}}_{\text{anti-commutador}} =$$

$$= \frac{1}{2}[\hat{A},\hat{B}] + \frac{1}{2}\{\Delta\hat{A},\Delta\hat{B}\} \qquad (8.3)$$

$$\Downarrow$$

$$[\hat{A},\hat{B}]^{+} = (\hat{A}\hat{B}-\hat{B}\hat{A})^{+} = \hat{B}^{+}\hat{A}^{+} - \hat{A}^{+}\hat{B}^{+} \underset{\uparrow}{=} -[A,B] \Rightarrow$$

$\hat{A},\hat{B}$ Hermitian

$[\hat{A},\hat{B}]$ is anti-Hermitian

$\Downarrow$

eigenvalues are imaginary $\Rightarrow \langle[\hat{A},\hat{B}]\rangle$ is imaginary

$$\{\Delta\hat{A},\Delta\hat{B}\}^{+} = (\Delta\hat{A}\Delta\hat{B} + \Delta\hat{B}\Delta\hat{A})^{+} \underset{\uparrow}{=} \Delta B\Delta A + \Delta A\Delta B =$$

$= \{\Delta\hat{A},\Delta\hat{B}\} \Rightarrow$ Hermitian $\qquad \Delta\hat{A},\Delta\hat{B}$ Hermitian

$\Downarrow$

eigenvalues are real

$\Downarrow$

Then, from (8.3) $\qquad \langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle$ is real

$\Downarrow$

$$\langle(\Delta\hat{A})(\Delta\hat{B})\rangle = \frac{1}{2}\underbrace{\langle[\Delta\hat{A},\Delta\hat{B}]\rangle}_{\text{imaginary}} + \frac{1}{2}\underbrace{\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle}_{\text{real}}$$

$$|\langle(\Delta\hat{A})(\Delta\hat{B})\rangle|^2 = \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2 + \frac{1}{4}|\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle|^2$$

$$\Downarrow$$

$$|\langle(\Delta\hat{A})(\Delta\hat{B})\rangle|^2 \geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2 \qquad (8.4)$$

Substitute (8.2) in (8.4) $\Rightarrow$

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2 \Rightarrow$$

$$(\Delta A)(\Delta B) \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle| \quad (8.5)$$

general form of the uncertainty relation

<u>Example</u> Heisenberg uncertainty relations

$\hat{A} = X$; $\hat{B} = P_x$

$[X, P_x] = i\hbar I$

identity operator

Apply (8.5) $\Rightarrow$ $\Delta X \cdot \Delta P_x \geq \frac{1}{2} |\langle i\hbar \rangle| = \frac{\hbar}{2}$

(Similarly $\Rightarrow$ $\Delta y \cdot \Delta P_y \geq \frac{\hbar}{2}$, $\Delta z \cdot \Delta P_z \geq \frac{\hbar}{2}$)

!

The position and momentum of a microscopic system cannot be accurately measured both at once $\Rightarrow$ if the position is measured with uncertainty ΔX, the uncertainty in measuring the momentum cannot be smaller than $\frac{\hbar}{2\Delta X}$!