Lecture #5

Properties of operators. Commutators

- The commutator of two operators A and B $\Rightarrow$

$$[A,B] = AB - BA$$

- The anti-commutator $\{A,B\} = AB + BA$
- Two operators commute if $[A,B] = 0$
- For any $A \Rightarrow [A,A] = 0$ is valid
- If two operators are Hermitian and their product is also Hermitian, these operators commute $\Rightarrow$

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} = BA \ , \ \text{since} \ \underset{\uparrow}{(AB)^{\dagger} = AB} \Rightarrow \underline{BA = AB}$$

i.e. product is Hermitian

Properties of commutators

- Antisymmetry $\Rightarrow [A,B] = -[B,A]$
- Linearity $\Rightarrow [A, B+C+D+\ldots] = [A,B] + [A,C] + [A,D] + \ldots$

Hermitian conjugate of a commutator $\Rightarrow$

$$[A,B]^{\dagger} = [B^{\dagger}, A^{\dagger}]$$

- Distributivity

$$[A, BC] = [A,B]C + B[A,C]$$

- Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

- Operators commute with scalars $\Rightarrow [A, a] = 0$
scalar

Functions of operators

Let $F(A)$ be a function of an operator A. If A is a linear operator, we can expand $F(A)$ in Taylor series:

$$F(A) = \sum_{n=0}^{\infty} a_n A^n$$

expansion coefficient

Example: scalar

$$e^{aA} = \sum_{n=0}^{\infty} \frac{a^n}{n!} A^n = I + aA + \frac{a^2}{2!} A^2 + \frac{a^3}{3!} A^3 \ldots$$

identity operator

- If $[A, B] = 0 \Rightarrow [B, F(A)] = 0$

operators

any function of an operator A

- $[F(A)]^+ = F^*(A^+) \Rightarrow$ example: $(e^{iA})^+ = e^{-iA^+}$

Note: if A is Hermitian, $F(A)$ is Hermitian <u>only</u> if F is a real function

Inverse and Unitary operators

If it exists (not all operators have inverse!), the inverse A^{-1} of a linear operator A is defined by $A^{-1}A = AA^{-1} = I$

unit operator (identity)

$$I|\psi\rangle = |\psi\rangle$$

eigenvalue of a unit operator = 1

What about an eigenvalue of A^{-1}? $\Rightarrow$

let's assume that we know that $A|\psi\rangle = a|\psi\rangle$

eigenvalue of A

We know that $\underbrace{A^{-1}A}_{=I}|\psi\rangle = |\psi\rangle$

$\Downarrow$

$$\underbrace{A^{-1}A}_{=I}|\psi\rangle = A^{-1}(\underbrace{A|\psi\rangle}_{=a|\psi\rangle}) = aA^{-1}|\psi\rangle = |\psi\rangle$$

$\Downarrow$

$$A^{-1}|\psi\rangle = \frac{1}{a}|\psi\rangle$$

if a is an eigenvalue of A $\Leftarrow$ then $\frac{1}{a}$ is an eigenvalue of A^{-1}

Note: in general, if $A|\psi\rangle = a|\psi\rangle \Rightarrow$

$F(A)|\psi\rangle = F(a)|\psi\rangle$, e.g. $e^{iA}|\psi\rangle = e^{ia}|\psi\rangle$

A linear operator is unitary if its inverse is equal to its adjoint: $U^{-1} = U^{+}$ $\Rightarrow$

$$\underset{\substack{\uparrow \\ \text{unitary} \\ \text{operator}}}{U}U^{+} = U^{+}U = I$$

A product of unitary operators is a unitary operator

$$(ABCD\ldots)(ABCD\ldots)^{+} = ABCD\ldots D^{+}C^{+}B^{+}A^{+}$$

$$= I$$

Theorem: The eigenvalues of unitary operators are complex numbers of moduli $= 1$; the eigenvectors of a unitary operator that has no degenerate eigenvalues are mutually orthogonal.

Proof: Let $U|\varphi_{n\,(m)}\rangle = a_{n\,(m)}|\varphi_{n\,(m)}\rangle$

$\uparrow$ eigenvalues of U

$$\langle\varphi_m|\underbrace{U^{+}U}_{= I}|\varphi_n\rangle = \langle\varphi_m|\varphi_n\rangle = a_m^{*}a_n\langle\varphi_m|\varphi_n\rangle$$

$$\Downarrow$$

$$(a_m^{*}a_n - 1)\langle\varphi_m|\varphi_n\rangle = 0$$

- $n = m \Rightarrow |a_n|^2 = 1 \Rightarrow |a_n| = 1$
- $n \neq m \Rightarrow \langle\varphi_m|\varphi_n\rangle = 0$

Projection operator. Matrix representation of kets, bras and operators

Consider a complete orthonormal set of the eigenkets of an operator A: $A|\varphi_n\rangle = a_n|\varphi_n\rangle$

$\langle\varphi_n|\varphi_m\rangle = \delta_{nm}$

An arbitiary ket $|\Psi\rangle$ can be presented as $|\Psi\rangle = \sum_n C_n|\varphi_n\rangle$,

with expansion coefficients $C_n = \langle\varphi_n|\Psi\rangle$

Then, $|\Psi\rangle = \underbrace{\sum_n |\varphi_n\rangle\langle\varphi_n|}_{= I\ (1)}\Psi\rangle \Rightarrow$ (identity operator)

$$\sum_n |\varphi_n\rangle\langle\varphi_n| = 1$$ ⇐ completeness relation or closure

Example: $|\varphi_1\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\varphi_2\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\sum_{n=1}^{2} |\varphi_n\rangle\langle\varphi_n| = |\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2| \doteq$$

$$\doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{matrix representation of } I \text{ in 2D space}} \doteq I$$

identity operator

Define the projection operator along the base ket $|\varphi_n\rangle$ as $P_n = |\varphi_n\rangle\langle\varphi_n|$

What does P_n project? $\Rightarrow$ act on $|\psi\rangle \Rightarrow$

$$P_n|\psi\rangle = |\varphi_n\rangle\underbrace{\langle\varphi_n|\psi\rangle}_{= C_n} = C_n|\varphi_n\rangle \Rightarrow$$

selected a portion of $|\psi\rangle$ "parallel" to $|\varphi_i\rangle$.

Properties of the projection operators

- $\sum_n P_n = 1 \Leftarrow$ completeness

- $P^{+} = P \Rightarrow$ Hermitian

Check: $\left(|\varphi_n\rangle\langle\varphi_n|\right)^{+} = |\varphi_n\rangle\langle\varphi_n|$

- $P^2 = P$

Check: $\underbrace{\left(|\varphi_n\rangle\langle\varphi_n|\right)}_{P_n}\underbrace{\left(|\varphi_n\rangle\langle\varphi_n|\right)}_{P_n} =$

$= |\varphi_n\rangle\underbrace{\langle\varphi_n|\varphi_n\rangle}_{1}\langle\varphi_n| = |\varphi_n\rangle\langle\varphi_n|$

$\leftarrow$ note that this is true only for orthonormal basis, otherwise $|\varphi_n\rangle\langle\varphi_n|$ is <u>not</u> a projector

- The product of two commuting projection operators, P_1 and P_2, is also a projection operator

Check: $(P_1P_2)^{+} = P_2^{+}P_1^{+} = P_2P_1 = P_1P_2 \Rightarrow$ Hermitian

$(P_1P_2)^2 = P_1P_2P_1P_2 = P_1^2P_2^2 = P_1P_2$

$\Rightarrow P_1P_2$ is a projection operator

- Two projection operators are orthogonal if their product is 0
- A sum of projectors $P_1 + P_2 + \ldots$ is generally not a projector unless $P_1, P_2, \ldots$ are mutually orthogonal.

try to prove this for yourself!

Projection operators are extremely useful $\Rightarrow$

Example: $|\Psi\rangle$ is an arbitrary ket

$\langle\Psi|\Psi\rangle$ - ? $\qquad |\Psi\rangle = \sum_n a_n |\varphi_n\rangle$

$$\langle\Psi|\Psi\rangle = \langle\Psi| \cdot \Big(\underbrace{\sum_n |\varphi_n\rangle\langle\varphi_n|}_{= I \text{ can insert}}\Big) \cdot |\Psi\rangle =$$

$$= \sum_n \langle\Psi|\varphi_n\rangle\langle\varphi_n|\Psi\rangle = \sum_n |\langle\Psi|\varphi_n\rangle|^2$$

If $|\Psi\rangle$ is normalized $\Rightarrow$ $\sum_n |\langle\Psi|\varphi_n\rangle|^2 =$

$$= \sum_n |a_n|^2 = 1$$