QM
Phys 651

Lecture #12

Momentum operator in the position basis

Consider momentum as a generator of translatio

(let's work in 1D for simplicity)

$$\hat{U}_{\Delta x} = \hat{I} - \frac{i}{\hbar}\hat{P}_x \Delta x \,, \quad \hat{U}_{\Delta x}|x\rangle = |x+\Delta x\rangle$$

Consider an arbitrary state $|\alpha\rangle$:

$$|\alpha\rangle = \int dx\, |x\rangle\langle x|\alpha\rangle \quad \leftarrow \text{expansion in terms of eigenbasis}$$

$$\hat{U}_{\Delta x}|\alpha\rangle = \int dx\, \underbrace{\hat{U}_{\Delta x}|x\rangle}_{=|x+\Delta x\rangle} \langle x|\alpha\rangle =$$

change of variables $x' \equiv x + \Delta x$

$$= \int dx'\, |x'\rangle\langle x' - \Delta x|\alpha\rangle =$$

$\nwarrow$ Taylor expansion

$$= \int dx'\, |x'\rangle \left(\langle x'|\alpha\rangle - \Delta x \frac{\partial}{\partial x'}\langle x'|\alpha\rangle \right) =$$

$$= |\alpha\rangle - \Delta x \int dx'\, |x'\rangle \frac{\partial}{\partial x'}\langle x'|\alpha\rangle \qquad (12.1)$$

Now go back to left-hand-side of (12.1):

$$\hat{U}_{\Delta x}|\alpha\rangle = (\hat{I} - \frac{i}{\hbar}\hat{P}_x \Delta x)|\alpha\rangle =$$

$$= |\alpha\rangle - \frac{i}{\hbar}\Delta x \hat{P}_x |\alpha\rangle \qquad (12.2)$$

Compare (13.1) and (12.2) $\Rightarrow$

$$\hat{P}_x|\alpha\rangle = -i\hbar\int dx'\,|x'\rangle\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

$$\langle x''|\hat{P}_x|\alpha\rangle = -i\hbar\int dx'\,\underbrace{\langle x''|x'\rangle}_{\delta(x''-x')}\frac{\partial}{\partial x'}\langle x'|\alpha\rangle =$$

$$= -i\hbar\frac{\partial}{\partial x''}\langle x''|\alpha\rangle \qquad (12.3)$$

If $|\alpha\rangle = |x'\rangle \Rightarrow \langle x''|\hat{P}_x|x'\rangle = -i\hbar\frac{\partial}{\partial x''}\langle x''|x'\rangle =$

$$= \underbrace{-i\hbar\frac{\partial}{\partial x'}\delta(x'-x'')}$$

matrix element of P_x in the x-representation

If $|\alpha\rangle, |\beta\rangle$ are arbitrary states $\Rightarrow$

$$\langle\beta|\hat{P}_x|\alpha\rangle = \int dx'dx''\langle\beta|x'\rangle\overbrace{\langle x'|\hat{P}_x|x''\rangle}^{= -i\hbar\frac{\partial}{\partial x'}\delta(x'-x'')}\langle x''|\alpha\rangle =$$

$$= \int dx' \langle \beta | x' \rangle \left(-i\hbar \frac{\partial}{\partial x'}\right) \langle x' | \alpha \rangle =$$

$$= \int dx' \, \psi_\beta^*(x') \left(-i\hbar \frac{\partial}{\partial x'}\right) \psi_\alpha(x')$$

Similarly, $\langle x' | P_x^n | \alpha \rangle = (-i\hbar)^n \frac{\partial^n}{\partial x'^n} \langle x' | \alpha \rangle$

$$\langle \beta | P_x^n | \alpha \rangle = \int dx' \, \psi_\beta^*(x') (-i\hbar)^n \cdot \frac{\partial^n}{\partial x'^n} \psi_\alpha(x')$$

Example

The system is in a state described by a real wave function $\psi(x)$. Find the expectation value of the momentum (in 1D case), i.e. $\langle \hat{P}_x \rangle$

$$\langle \hat{P} \rangle = \langle \psi | \hat{P} | \psi \rangle = \int_{-\infty}^{+\infty} dx' dx'' \langle \psi | x' \rangle \langle x' | \hat{P} | x'' \rangle \cdot$$

$$\cdot \langle x'' | \psi \rangle = \int_{-\infty}^{+\infty} \psi^*(x') \left(-i\hbar \frac{d}{dx'} \delta(x' - x'')\right) \psi(x'') dx' dx$$

$$= -i\hbar \int_{-\infty}^{+\infty} dx' \, \psi^*(x') \frac{d}{dx'} \psi(x') \underset{\substack{\uparrow \\ \text{integrate} \\ \text{by parts}}}{=} -i\hbar \Big(\underbrace{\psi^* \psi \Big|_{-\infty}^{+\infty}}_{\substack{= 0 \\ \text{for a well-behaved function}}} -$$

$$-\int_{-\infty}^{+\infty} dx'\, \psi(x') \frac{d\psi^*(x')}{dx'} = i\hbar \int_{-\infty}^{+\infty} dx'\, \psi(x') \frac{d\psi^*(x')}{dx'}$$

$$\Downarrow$$

$$-i\hbar \int_{-\infty}^{+\infty} dx'\, \psi^*(x') \frac{d\psi(x')}{dx'} = i\hbar \int_{-\infty}^{+\infty} dx'\, \psi(x') \frac{d\psi^*(x')}{dx'}$$

$$\Downarrow$$

Since our $\psi(x)$ is real $\Rightarrow$ $\psi = \psi^*$

$$\Downarrow$$

$$\langle P \rangle = 0$$

Momentum-space wave function

Consider $|p\rangle$ - basis (for simplicity $\Rightarrow$ 1D case)

Similarly to the position space $\Rightarrow$

$$\hat{P}|p'\rangle = p'|p'\rangle \; ; \quad \underset{\text{orthogonality}}{\langle p'|p''\rangle = \delta(p'-p'')}$$

Expansion of an arbitrary state $|\alpha\rangle \Rightarrow$

$$|\alpha\rangle = \int dp' |p'\rangle\langle p'|\alpha\rangle$$

Probability that a measurement of P yields an eigenvalue p' within an interval $\Delta p = dp'$ is $|\langle p'|\alpha\rangle|^2 \underbrace{dp'}_{\Delta p}$ (similar to the measurement of X - see Lecture #12)

$\langle p'|\alpha\rangle = \varphi_\alpha(p')$ ← momentum-space wave function

If $|\alpha\rangle$ is normalized $\Rightarrow \int dp' |\langle p'|\alpha\rangle|^2 =$

$$= \int dp' |\varphi_\alpha(p')|^2 = 1$$

What is the connection between x- and p-representation

Recall: $\langle x''|\hat{p}|\alpha\rangle = -i\hbar \frac{\partial}{\partial x''}\langle x''|\alpha\rangle$
(13.3)

Let $|\alpha\rangle = |p'\rangle$ ← momentum eigenket

$$\langle x''|\underbrace{\hat{p}|p'\rangle}_{= p'|p'\rangle} = -i\hbar \frac{\partial}{\partial x''}\langle x''|p'\rangle \Rightarrow$$

$$p'\langle x''|p'\rangle = -i\hbar \frac{\partial}{\partial x''}\langle x''|p'\rangle$$

↑ differential equation for $\langle x''|p'\rangle$

$$p' f = -i\hbar \frac{\partial}{\partial x''} f \Rightarrow f = N e^{\frac{i}{\hbar}p'x''}$$

say, $f(x'', p')$

Drop "primes" for simplicity $\Rightarrow \langle x|p\rangle = N e^{\frac{i}{\hbar}px}$

prob. amplitude for the momentum eigenstate $|p\rangle$ to be found at position x

normaliz. const

Normalization: $\langle x'|x''\rangle = \delta(x'-x'') =$

$$= \int dp' \langle x'|p'\rangle\langle p'|x''\rangle =$$

$$= |N|^2 \int dp'\, e^{\frac{i}{\hbar}p'(x'-x'')} \underset{\uparrow}{=} |N|^2 \cdot 2\pi\hbar\,\delta(x'-x'') \Rightarrow$$

Recall

$$\Rightarrow \underset{\uparrow}{|N|} = \frac{1}{\sqrt{2\pi\hbar}} \qquad \delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-x_0)}\,dk$$

say, real

$\Downarrow$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{ipx}{\hbar}} \leftarrow \text{plane wave}$$

How is a position-space wave function $\psi_\alpha(x)$ related to a momentum-space one $\phi_\alpha(p)$? $\Rightarrow$

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \int dp\, \langle x|p\rangle \underbrace{\langle p|\alpha\rangle}_{=\phi_\alpha(p)} =$$

$$= \int dp \cdot \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{ipx}{\hbar}}\, \phi_\alpha(p)$$

$$\phi_\alpha(p) = \langle p|\alpha\rangle = \int dx \langle p|x\rangle\langle x|\alpha\rangle = \int dx \cdot \frac{1}{\sqrt{2\pi\hbar}}\, e^{-\frac{i}{\hbar}px}\, \psi_\alpha(x)$$

$\Downarrow$

$\psi_\alpha(x)$ and $\phi_\alpha(p)$ are Fourier transforms of each other!