QM
Phys 651

Solution of HW #4

Problem #1

$$H = \mathcal{E}_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; \quad |\Psi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

(a) This is basically an eigenvalue problem

Diagonalise H $\Rightarrow$ $\det \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix} = 0 \Rightarrow$

$$-(\lambda+1)\left[(1-\lambda)^2 - 1\right] = 0 \Rightarrow \quad 1)\ \lambda_1 = -1$$
$$2)\ 1-\lambda = \pm 1$$
$$\lambda_{2;3} = \begin{cases} 0 \\ 2 \end{cases}$$

So, possible outcomes of the measurements are

$$E_1 = -\mathcal{E}_0$$
$$E_2 = 0$$
$$E_3 = 2\mathcal{E}_0$$

To find probabilities, need to find eigenvectors:

$|E_1\rangle$: $(\lambda_1 = -1)$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0 \Rightarrow C_3 - \text{arbitrary}$$

$$\begin{cases} 2C_1 = C_2 \\ -C_1 = -2C_2 \end{cases}$$

$$|E_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Leftarrow \text{true if } C_1 = C_2 = 0$$

(α)

$|E_2\rangle$: $(\lambda_2 = 0)$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \Rightarrow c_3 = 0, \quad c_1 = c_2 =$$

$$|E_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$|E_3\rangle$: $(\lambda_3 = 2)$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \Rightarrow c_3 = 0, \quad c_1 = -c_2$$

$$\Downarrow$$

$$|E_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

The probability to end up in $|E_1\rangle$:

$$P_1 = |\langle\Psi|E_1\rangle|^2 = \left|\frac{1}{\sqrt{6}} (1\ 1\ 2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = \left(\frac{2}{\sqrt{6}}\right)^2 = \frac{2}{3}$$

Similarly:

$$P_2 = |\langle\Psi|E_2\rangle|^2 = \left|\frac{1}{\sqrt{6}} (1\ 1\ 2) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right|^2 = \frac{4}{12} = \frac{1}{3}$$

$$P_3 = |\langle\Psi|E_3\rangle|^2 = \left|\frac{1}{\sqrt{6}} (1\ 1\ 2) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right|^2 = 0$$

Check: $P_1 + P_2 + P_3 = 1$

(Note: Initial state $|\Psi\rangle$ can be presented as $|\Psi\rangle = \frac{2}{\sqrt{6}} |E_1\rangle + \frac{1}{\sqrt{3}} |E_2\rangle + 0 \cdot |E_3\rangle$, from which one could directly read out the probabilities)

(b) Expectation value can be found by two ways: ③

(i) $\langle H \rangle = \mathcal{P}_1 E_1 + \mathcal{P}_2 E_2 + \mathcal{P}_3 E_3 =$

$= \frac{2}{3} \cdot (-\mathcal{E}_0) + \frac{1}{3} \cdot 0 + 0 \cdot 2\mathcal{E}_0 = -\frac{2}{3}\mathcal{E}_0$

(ii) $\langle H \rangle = \langle \Psi | H | \Psi \rangle = \frac{\mathcal{E}_0}{6} \begin{pmatrix} 1 & 1 & 2 \end{pmatrix} \cdot$

$\cdot \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{\mathcal{E}_0}{6} \begin{pmatrix} 1 & 1 & 2 \end{pmatrix} \cdot$

$\cdot \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = -\frac{4}{6}\mathcal{E}_0 = -\frac{2}{3}\mathcal{E}_0$

Problem # 2

$$H = \varepsilon_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad A = a_0 \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) Measure energy $\Rightarrow$ get eigenvalues of H;

$$\det \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix} = 0 \Rightarrow -(1-\lambda)^2(1+\lambda) + \lambda + 1 = 0;$$

1) $\lambda = -1 \Rightarrow E = -\varepsilon_0$

2) $\lambda = 0 \Rightarrow E = 0$

3) $\lambda = 2 \Rightarrow E = 2\varepsilon_0$

(b) Let's find the state of the system after measuring $E = -\varepsilon_0$;

$$|E = -\varepsilon_0\rangle : \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \Rightarrow$$

$$\underbrace{|E = -\varepsilon_0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{} \Leftarrow \begin{array}{l} c_3 \text{ arbitrary} \\ c_2 = 2c_1 = \frac{c_1}{2} = 0 \end{array}$$

This is the state right before our measurement of A

Eigenvalues of A (results of measurements): (9)

$$\det \begin{pmatrix} -\lambda & 4 & 0 \\ 4 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix} = 0 \Rightarrow -\lambda(\lambda^2-1) - 4(-4\lambda) = 0$$

$$\Downarrow$$

1) $\lambda_1 = 0 \Rightarrow \underline{A = 0}$

2) $\lambda^2 - 1 - 16 = 0$

$\lambda_{2,3} = \pm\sqrt{17} \Rightarrow$

$\underline{A = \pm\sqrt{17}\, a_0}$

Eigenvectors:

$|A=0\rangle$: $\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \Rightarrow c_2 = 0$, $c_3 = -4c_1$

$$\Downarrow$$

$$\underline{\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \cdot \frac{1}{\sqrt{17}}}$$

$|A=\sqrt{17}\,a_0\rangle$: $\begin{pmatrix} -\sqrt{17} & 4 & 0 \\ 4 & -\sqrt{17} & 1 \\ 0 & 1 & -\sqrt{17} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$

$$\Downarrow$$

$$4c_2 = \sqrt{17}\, c_1, \quad c_2 = \sqrt{17}\, c_3$$

$$\Downarrow$$

$$\underline{\begin{bmatrix} 4 \\ \sqrt{17} \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{34}}}$$

$$|A = -\sqrt{17}a_0 > : \begin{pmatrix} \sqrt{17} & 4 & 0 \\ 4 & \sqrt{17} & 1 \\ 0 & 1 & \sqrt{17} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

$$\Downarrow$$

$$\begin{bmatrix} 4 \\ -\sqrt{17} \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{34}}$$

Probabilities:

- for an outcome $A = 0$:

$$\mathcal{P}(A=0) = |\langle A=0 | E = -\varepsilon_0 \rangle|^2 =$$

$$= \left| \frac{1}{\sqrt{17}} [1 \; 0 \; -4] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{16}{17}$$

- for an outcome $A = \sqrt{17}a_0$:

$$\mathcal{P}(A=\sqrt{17}a_0) = |\langle A = \sqrt{17}a_0 | E = -\varepsilon_0 \rangle|^2 =$$

$$= \left| \frac{1}{\sqrt{34}} [4 \; \sqrt{17} \; 1] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{34}$$

- for an outcome $A = -\sqrt{17}a_0$:

$$\mathcal{P}(A=-\sqrt{17}a_0) = |\langle A = -\sqrt{17}a_0 | E = -\varepsilon_0 \rangle|^2 = \frac{1}{34}$$

Check: $\mathcal{P}(A=0) + \mathcal{P}(A=\sqrt{17}a_0) + \mathcal{P}(A=-\sqrt{17}a_0) = \underset{\sim}{1}$

Expectation value:

$$\langle A \rangle = \mathcal{P}(A=0)\cdot 0 + \mathcal{P}(A=\sqrt{17}a_0)\cdot\sqrt{17}a_0 + \mathcal{P}(A=-\sqrt{17}a_0)\cdot(-\sqrt{17}a_0) = \frac{1}{34}\cdot\sqrt{17}a_0 + \frac{1}{34}\cdot(-\sqrt{17}a_0) = 0$$

Problem #1

$$|\psi\rangle \doteq \frac{1}{6}\begin{pmatrix}1\\0\\4\end{pmatrix}; \quad A \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & i\\ 0 & -i & 1\end{pmatrix};$$

$$B \doteq \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}$$

(a) Measure $A \Rightarrow$ let's find eigenvalues $\Rightarrow$

$$\det\begin{pmatrix}2-\lambda & 0 & 0\\ 0 & 1-\lambda & i\\ 0 & -i & 1-\lambda\end{pmatrix} = 0 \Rightarrow$$

1) $\lambda_1 = 2$

2) $(1-\lambda)^2 - 1 = 0$

$\lambda_2 = 0$;

$\lambda_3 = 2 \leftarrow$ doubly-degenerate

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Eigenstates ~~for~~ $|\lambda_{(A)} = 0\rangle$:

$\uparrow$ observable

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 + i c_3 = 0 \Rightarrow c_2 = -i c_3 \end{array}$$

$$|\lambda_{(A)} = 0\rangle \doteq \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

Probability to obtain $A = 0$:

$$\mathcal{P}(A=0) = \frac{|\langle \lambda_{(A)} = 0 | \psi \rangle|^2}{\underbrace{\langle \psi | \psi \rangle}_{= \frac{17}{36}}} = \frac{36}{17} \left| \frac{1}{\sqrt{2}} [0 \; i \; 1] \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \frac{1}{6} \right|^2$$

$$= \frac{\cancel{36}}{17} \frac{1}{2} \frac{1}{\cancel{36}} \cdot 16 = \frac{8}{17}$$

Now measure $B \Rightarrow$ find eigenvalues $\Rightarrow$

$$\det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -i \\ 0 & i & -\lambda \end{bmatrix} = 0 \Rightarrow \begin{array}{l} \lambda = 1 \leftarrow \text{doubly-degener.} \\ \lambda = -1 \end{array}$$

Let's find an eigenstate $|\lambda_{(B)}=-1\rangle$;

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} C_1 = 0 \\ C_2 = iC_3 \end{matrix} \Rightarrow \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

Probability to find $B=-1$ is

$$\mathcal{P}(B=-1) = |\langle \lambda_{(B)}=-1|\lambda_{(A)}=0\rangle|^2 =$$

$$= \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}\cdot\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{4}|-1+1|^2 = 0$$

Then, probability to find $B=1$ is $1-\mathcal{P}(B=-1) = \underline{1}$

Alternatively,

$$\mathcal{P}(B=1) = |\langle \lambda_{(B)}=1|\lambda_{(A)}=0\rangle|^2$$

However, $B=1$ is doubly-degenerate $\Rightarrow$ need to find two eigenvectors $|\lambda_{(B)}=1;1\rangle$ and $|\lambda_{(B)}=1;2\rangle$.

$|\lambda_{(B)}=1\rangle$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -i \\ 0 & i & -1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} C_1 \text{ - arbitrary} \\ C_2 = -iC_3 \end{matrix} \Rightarrow$$

$$|\lambda_{(B)}=1;1\rangle = \begin{bmatrix}0\\-i\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}};$$

$$|\lambda_{(B)}=1;2\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\mathcal{P}(B=1) = |\langle\lambda_{(B)}=1;1|\lambda_{(A)}=0\rangle|^2 +$$

$$+|\langle\lambda_{(B)}=1;2|\lambda_{(A)}=0\rangle|^2 =$$

$$= \left|\frac{1}{\sqrt{2}}[0\ i\ 1]\begin{bmatrix}0\\-i\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\right|^2 + \left|[1\ 0\ 0]\begin{bmatrix}0\\-i\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\right|^2 =$$

$$= \frac{1}{4}|1+1|^2 + 0 = \underset{\sim}{1}$$

So, when B is measured <u>after</u> A, the result will be $B=1$ with certainty.

(b) Now let's first measure B $\Rightarrow$

$$\mathcal{P}(B=1) = \frac{|\langle\lambda_{(B)}=1;1|\psi\rangle|^2}{\langle\psi|\psi\rangle} + \frac{|\langle\lambda_{(B)}=1;2|\psi\rangle|^2}{\langle\psi|\psi\rangle} =$$

$$= \frac{36}{17}\cdot\left(\left|\frac{1}{\sqrt{2}}[0\ i\ 1]\begin{bmatrix}1\\0\\4\end{bmatrix}\frac{1}{6}\right|^2 + \left|[1\ 0\ 0]\begin{bmatrix}1\\0\\4\end{bmatrix}\frac{1}{6}\right|^2\right) =$$

$$= \frac{36}{17}\left(\frac{1}{2}\cdot\frac{1}{36}\cdot 16 + \frac{1}{36}\right) = \frac{8}{17} + \frac{1}{17} = \frac{9}{17}$$

Now measure $A = 0$:

$$\mathcal{P}(A=0) = |\langle \lambda_{(A)} = 0 \mid \lambda_{(B)} = 1 \rangle|^2$$

How do we know $|\lambda_{(B)} = 1\rangle$? Is it $|\lambda_{(B)} = 1; 1\rangle$, $|\lambda_{(B)} = 1; 2\rangle$ or their combination? $\Rightarrow$

new state is $|\chi\rangle = |\lambda_{(B)} = 1; 1\rangle\langle \lambda_{(B)} = 1; 1 | \psi\rangle +$

$$+ |\lambda_{(B)} = 1; 2\rangle\langle \lambda_{(B)} = 1; 2 | \psi\rangle =$$

$$= \frac{2}{3\sqrt{2}} |\lambda_{(B)} = 1; 1\rangle + \frac{1}{6} |\lambda_{(B)} = 1; 2\rangle$$

$$\mathcal{P}(A=0) = \frac{|\langle \lambda_{(A)} = 0 | \chi\rangle|^2}{\underbrace{\langle \chi | \chi \rangle}_{= \frac{4}{18} + \frac{1}{36} = \frac{9}{36} = \frac{1}{4}}} = 4 \cdot \left| \frac{1}{\sqrt{2}} [0 ; 1] \right. .$$

$$\left(\cancel{\frac{2}{3\sqrt{2}}} \cdot \cancel{\frac{1}{\sqrt{2}}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)\Bigg|^2 = \frac{4}{2}\left|\frac{2}{3}\right|^2 = \frac{8}{9}$$

(c) So, probability to obtain $A=0$ and <u>then</u> $B=1$ is

$$\mathcal{P}(A=0, B=1) = \mathcal{P}(A=0)\,\mathcal{P}(B=1) =$$

$$= \frac{8}{17} \cdot 1 = \frac{8}{17}$$

Probability to obtain $B=1$ and then $A=0$ is

$$\mathcal{P}(B=1, A=0) = \mathcal{P}(B=1)\,\mathcal{P}(A=0) = \frac{9}{17} \cdot \frac{8}{9} =$$

$$= \frac{8}{17} = \mathcal{P}(A=0, B=1)$$

So, in this case it doesn't matter whether A or B is measured first. Is this expected? $\Rightarrow$ Check whether A and B commute:

$$[A,B] = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} - \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = 0 \quad \checkmark \quad \underline{\underline{\text{Yes}}}$$

Problem #5

$$A \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad B \doteq \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ -1 & -i & 4 \end{pmatrix}; \quad C \doteq \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$$

(a) A: $\det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} = 0 \Rightarrow$ 1) $\lambda = 1$

2) $\lambda = \pm 1$

$\Downarrow$

$\underline{A = -1,\ 1}$ ← doubly-degenerate

B: $\det \begin{bmatrix} -\lambda & 0 & -1 \\ 0 & -\lambda & i \\ -1 & -i & 4-\lambda \end{bmatrix} = 0 \Rightarrow -\lambda(-\lambda(4-\lambda)-1) + {}$

$+\lambda = 0$;

$\Downarrow$

1) $\lambda = 0$

2) $\lambda(4-\lambda) + 1 + 1 = 0$

$\lambda^2 - 4\lambda - 2 = 0$

$D = 16 + 4 \cdot 2 = 24$

$\lambda = \dfrac{4 \pm 2\sqrt{6}}{2} = \underline{2 \pm \sqrt{6}}$

$\underline{B = 0,\ 2 \pm \sqrt{6}}$ ⇐

all (non-degenerate)

C: $\det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 3 & 1-\lambda \end{bmatrix} = 0 \Rightarrow$ 1) $\lambda = 2$

2) $(1-\lambda)^2 = 9$, $1-\lambda = \pm 3$

$\underline{C = -2, 2, 4}$ ⇐ (all non-deg.) $\lambda = 4;\ -2$

(b) Compatibility check:

$$[A,B] \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ -1 & -i & 4 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ -1 & -i & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ -1 & -i & 4 \\ 0 & 0 & i \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & i & 0 \\ -1 & 4 & -i \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -2i & 4 \\ 1 & -4 & 2i \end{pmatrix} \neq 0 \;\Downarrow$$

A & B are incompatible

B & C are incompatible

$$[B,C] \doteq \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ -1 & -i & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ -1 & -i & 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -3 & -1 \\ 0 & 3i & i \\ -2 & -i+12 & -3i+4 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -2 \\ -3 & -3i & i+12 \\ -1 & -i & 3i+4 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 1 \\ 3 & 6i & -12 \\ -1 & 12 & -6i \end{pmatrix}$$

$$[A,C] \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} = 0 \leftarrow$$

A, C are compatible

Find a basis of eigenvectors common to A & C:

A: $|A=-1\rangle$:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = -c_3 \end{matrix}$$

$$\Downarrow$$

$$|A=-1\rangle \doteq \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\cdot\frac{1}{\sqrt{2}}}\ \text{or}\ \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\cdot\frac{1}{\sqrt{2}}}$$

$|A=1\rangle$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} c_1 - \text{arbitrary} \\ c_2 = c_3 \end{matrix}$$

$$\Downarrow$$

$$|A=1;1\rangle \doteq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|A=1;2\rangle \doteq \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\cdot\frac{1}{\sqrt{2}}$$

C: $|C=-2\rangle$:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} c_1 = 0 \\ 3c_2 = -3c_3 \end{matrix}$$

$$\Downarrow$$

$$|C=-2\rangle \doteq \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\cdot\frac{1}{\sqrt{2}}}$$

$|C=2\rangle$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow c_1 - \text{arb.};\ \left.\begin{matrix} c_2 = 3c_3 \\ 3c_2 = c_3 \end{matrix}\right| \Rightarrow \begin{matrix} c_2 = 0 \\ c_3 = 0 \end{matrix}$$

$$|C=2\rangle \doteq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Leftarrow$$

$|C=4\rangle$: $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow$ $c_1 = 0$, $c_2 = c_3$

$$|C=4\rangle \doteq \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

So, a basis common for A & C is:

$$|A=-1, C=-2\rangle \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$|A=1, C=2\rangle \doteq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|A=1, C=4\rangle \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(c) Since A has degenerate eigenvalues $\Rightarrow$
$\{A\}$ is <u>not</u> a C.S.C.O

B, C do not have degenerate eigenvalues $\Rightarrow$
$\{B\}$ <u>is</u> a C.S.C.O
$\{C\}$ <u>is</u> a C.S.C.O

$\{A, B\}$ is <u>not</u> a C.S.C.O since $[A, B] \neq 0$

$\{B, C\}$ is <u>not</u> a C.S.C.O since $[B, C] \neq 0$

$\{A, C\}$ <u>is</u> a C.S.C.O