PH 585
AMO

Lecture #4

①

Spontaneous emission of radiation

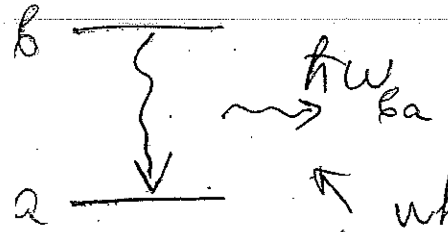
First-order transition rate for emission:

$$\overline{W}_{ab} = \frac{\pi\hbar}{\varepsilon_0}\left(\frac{e}{m}\right)^2 \frac{\overset{\text{stimulated}}{N(\omega_{ba})} + \overset{\text{spontaneous}}{1}}{V\omega_{ba}} |M_{ba}|^2$$

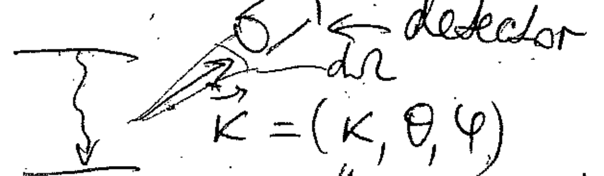
If $N = 0$ ← no incident radiation ⟹

$$\overline{W}_{ab}^{s} \leftarrow \text{spont. em.}$$

$$= \frac{\pi}{\varepsilon_0}\left(\frac{e}{m}\right)^2 \frac{\hbar}{\underset{\text{volume} = L^3}{V}\omega_{ba}} |M_{ba}|^2 \,\delta(\omega - \omega_{ba})$$ ← not to forget assumptions



where does the photon go?



$\vec{K} = (k, \theta, \varphi)$, $k = \frac{\omega}{c}$

$$d\vec{k} = k^2\,dk\,d\Omega = dk_x\,dk_y\,dk_z$$

Number of states available for emitted photons ⟹

$$k_{x,y,z} = \frac{2\pi}{L} n_{x,y,z}$$

$$dn = dn_x\,dn_y\,dn_z = \left(\frac{L}{2\pi}\right)^3 \underbrace{k^2\,dk}_{\frac{\omega^2}{c^3}d\omega}\,d\Omega$$

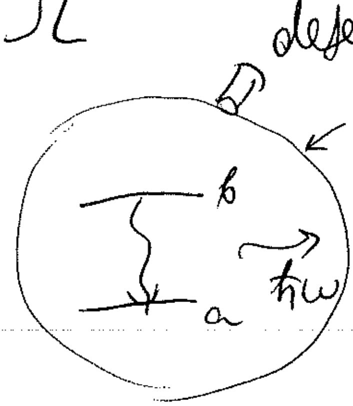
$$\underbrace{\overline{W}^{s}_{ab}(\theta,\varphi)\,d\Omega}_{\text{emission of a photon into } d\Omega \text{ of } (\theta,\varphi)} = \int_{\omega} \overline{W}^{s}_{ab}\, dn = \left(\frac{L}{2\pi}\right)^3 \frac{\omega_{ba}^2}{c^3}.$$

$$\int_0^\infty \delta(\omega-\omega_{ba})\,\omega^2\,d\omega$$

$$\cdot\ \frac{\pi}{\varepsilon_0}\left(\frac{e}{m}\right)^2 \frac{\hbar}{\cancel{V}\omega_{ba}} |M_{ba}|^2\, d\Omega = \frac{\hbar}{(2\pi)^3}\frac{\pi}{\varepsilon_0}\left(\frac{e}{m}\right)^2 \frac{\omega_{ba}}{c^3}.$$

$$\cdot\ |M_{ba}|^2\, d\Omega$$

What if



integrating sphere ← pick up all photons emitted in all directions but with a specific polarization $\vec{\varepsilon}$

$\Downarrow$

$$W^{s}_{ab} = \frac{\hbar}{(2\pi)^3}\frac{\pi}{\varepsilon_0}\left(\frac{e}{m}\right)^2 \frac{\omega_{ba}}{c^3}\int d\Omega\, |M_{ba}|^2$$

If take into account all polarizations $\Rightarrow$

$$|M_{ba}|^2 \Rightarrow \sum_{\lambda=1}^{2} |M^{\lambda}_{ba}|^2 \Leftarrow M^{\lambda}_{ba} = \langle \psi_b | e^{i\vec{k}\cdot\vec{r}}\, \vec{\varepsilon}_\lambda \cdot \vec{\nabla} |\psi_a\rangle$$

In the electric dipole approximation $\Rightarrow$

$$W^{s,D}_{ab} = \frac{\hbar}{(2\pi)^3}\frac{\pi}{\varepsilon_0}\left(\frac{e}{m}\right)^2 \frac{\omega_{ba}}{c^3}\cdot\frac{m^2\omega_{ba}^2}{\hbar^2 e^2}\int |\hat{\varepsilon}\cdot \vec{D}_{ba}|^2\, d\Omega$$

$-e\vec{r}_{ba}$

polariz. $\hat{\varepsilon}$

All polarizations: $|\hat{\vec{\varepsilon}}\cdot\vec{D}_{ba}|^2 \to |\hat{\vec{\varepsilon}}_1\vec{D}_{ba}+\hat{\vec{\varepsilon}}_2\vec{D}_{ba}|^2$ 3

$$w_{ab}^{s,D} = \frac{4}{3\hbar c^3}\frac{1}{4\pi\varepsilon_0}\omega_{ba}^3|\vec{D}_{ba}|^2$$

HW!

Example: Spontaneous emission $2p \to 1s$ of a hydrogenic atom in the electric dipole approximation

$l=1$, $m=0,\pm1$

E_{2p}

$$\omega_{ba} = \frac{E_b - E_a}{\hbar} = -Z^2E_I\cdot\left[\frac{1}{2^2}-1\right] = \frac{3}{4}Z^2\underbrace{E_I}_{\frac{e^2}{2a_0(4\pi\varepsilon_0)}}$$

E_{1s}; E_I: ioniz. energy of (H)

$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$, $a_\mu = a_0\frac{m}{\mu}$

a_0 or, more precisely, a_μ

Need $|\underbrace{\vec{D}_{ba}}_{-e\vec{r}_{ba}}|^2$

$$\vec{r}_{ba} = \langle\Psi_{2p}|\vec{r}|\Psi_{1s}\rangle = \int_0^\infty R_{21}(r)R_{10}(r)r^3dr\,.$$

$$\cdot\int Y_{1m}^*(\theta,\varphi)\underbrace{\begin{Bmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{Bmatrix}}_{\hat{\vec{r}}\ \leftarrow\ \text{unit vector}}\underbrace{Y_{00}(\theta,\varphi)}_{=\frac{1}{\sqrt{4\pi}}}d\Omega$$

$$\sin\theta\cos\varphi = \sqrt{\frac{2\pi}{3}}\left[-Y_{1,1}(\theta,\varphi)+Y_{1,-1}(\theta,\varphi)\right]$$

$$\sin\theta\sin\varphi = \sqrt{\frac{2\pi}{3}}\,i\left[Y_{1,1}(\theta,\varphi)+Y_{1,-1}(\theta,\varphi)\right]$$

$$\cos\theta = \sqrt{\frac{4\pi}{3}}Y_{1,0}(\theta,\varphi)$$

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Then, $\int Y^*_{1m}(\theta,\varphi)\, Y_{1m'}(\theta,\varphi)\, d\Omega = \delta_{mm'}$

$$\vec{r}_{ba} = \underbrace{\int_0^\infty R_{21}(r) R_{10}(r)\, r^3\, dr}_{\parallel} \cdot \frac{1}{\sqrt{4\pi}} \begin{cases} \sqrt{\frac{2\pi}{3}}\,(-\delta_{m,1} + \delta_{m,-1}) \\ \sqrt{\frac{2\pi}{3}}\, i\, (\delta_{m,1} + \delta_{m,-1}) \\ \sqrt{\frac{4\pi}{3}}\, \delta_{m,0} \end{cases}$$

$$\frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{3/2} \frac{Z}{a_0} \cdot 2\left(\frac{Z}{a_0}\right)^{3/2} \underbrace{\int_0^\infty e^{-\frac{3}{2}\frac{Zr}{a_0}}\, r^4\, dr}_{\left(\frac{2a_0}{3Z}\right)^5 \cdot \underbrace{\Gamma(5)}_{=4!}} = \frac{a_0}{Z}\left(\frac{2}{3}\right)^5 \frac{24}{\sqrt{6}}$$

$$|\vec{r}_{ba}|^2 = \left(\frac{a_0}{Z}\right)^2 \left(\frac{2}{3}\right)^{10} \underbrace{\frac{24^2}{6}}_{24\cdot 4} \cdot \frac{1}{4\pi}\left[\frac{2\pi}{3}\overbrace{(-\delta_{m,1} + \delta_{m,-1})^2}^{=\delta_{m,1}+\delta_{m,-1}} + \right.$$

$$\left. + \frac{2\pi}{3}\underbrace{(\delta_{m,1} + \delta_{m,-1})^2}_{\delta_{m,1}+\delta_{m,-1}} + \frac{4\pi}{3}\underbrace{\delta_{m,0}^2}_{\delta_{m,0}}\right] = \left(\frac{a_0}{Z}\right)^2 \frac{2^{14}}{3^{10}} \cdot$$

$$\cdot\left[\delta_{m,1} + \delta_{m,-1} + \delta_{m,1} + \delta_{m,-1} + 2\delta_{m,0}\right] = \left(\frac{a_0}{Z}\right)^2 \frac{2^{15}}{3^{10}} \cdot$$

$\cdot \left[\delta_{m,1} + \delta_{m,-1} + \delta_{m,0}\right] \leftarrow$ same probability of transition for any m, i.e. $1, 0, -1$

$$W_{ab}^{s,D} = \underbrace{\frac{4}{3\hbar c^3}\frac{1}{4\pi\varepsilon_0}\omega_{ba}^3 \cdot e^2 \cdot \left(\frac{a_0}{Z}\right)^2 \frac{2^{15}}{3^{10}}}_{=C'} \left[\delta_{m,1} + \delta_{m,-1} + \delta_{m,0}\right]$$

$$= \frac{1}{3}\sum_{m=-1}^{1}\left[\delta_{m,1} + \delta_{m,-1} + \delta_{m,0}\right] \cdot C' = C' =$$

If 210, 211, 21-1 levels are equally populated

Show

$$= \left(\frac{2}{3}\right)^8 \frac{m\alpha^5 Z^4 c^2}{\hbar} = 6{,}27 \cdot 10^8 Z^4 s^{-1}$$

$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137} \leftarrow$ fine structure constant